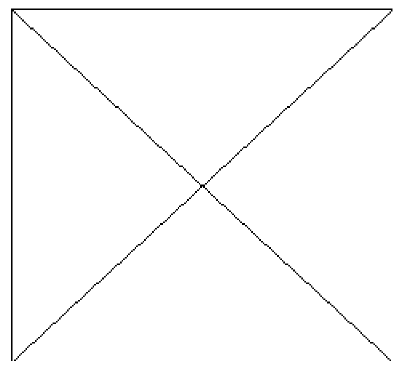
Test of Ingenuity



Bay Area Math Meet
University of San Francisco
April 28, 2001

Directions: Please do not open the test booklet until the proctor begins the examination. This is a multiple-choice, 20-question exam. You will have one hour to work on the problems. You get one point for a correct answer, zero points for no answer, and $-1/4$ points for an incorrect answer. Because of this penalty for guessing, if you are not sure of the correct answer to a question, it is best not to answer the question.

The questions are arranged in roughly increasing order of difficulty. The last 6 questions are particularly hard and will be used to break ties. Unless you are extremely ambitious, do not attempt all of the problems! Answering just half of them correctly is a very fine achievement! Read the questions first to see which problems are best for you.

Good luck, and have fun!

1 Find the circumference of a circle with area 36π. $\boxed{12\pi}$

$\boxed{\text{A}}$ 6π $\boxed{\text{B}}$ 12π $\boxed{\text{C}}$ 36π $\boxed{\text{D}}$ 72π $\boxed{\text{E}}$ 324π

2 If $x + y = 5$ and $xy = 6$, compute $\boxed{\mathbf{5/6}}$

$$\frac{1}{x} + \frac{1}{y}.$$

$\boxed{\text{A}}$ $\frac{1}{6}$ $\boxed{\text{B}}$ $\frac{1}{2}$ $\boxed{\text{C}}$ $\frac{1}{3}$ $\boxed{\text{D}}$ $\frac{5}{6}$ $\boxed{\text{E}}$ $\frac{6}{5}$

3 The length, width, height, and longest diagonal of a rectangular box are integers. What is the smallest possible value for the length of the longest diagonal of the box? $\boxed{\mathbf{3}}$

$\boxed{\text{A}}$ 2 $\boxed{\text{B}}$ 3 $\boxed{\text{C}}$ 4 $\boxed{\text{D}}$ 5 $\boxed{\text{E}}$ 7

4 Anna has two unmarked water bottles. One holds 12 ounces of fluid and the other holds 42 ounces. Anna has access to running water, but has no other water containers. What is the smallest nonzero amount of water that Anna can exactly measure out? **6**

$\boxed{\text{A}}$ 4 $\boxed{\text{B}}$ 5 $\boxed{\text{C}}$ 6 $\boxed{\text{D}}$ 7 $\boxed{\text{E}}$ 8

5 A $3 \times 3 \times 3$ cube is built out of 27 $1 \times 1 \times 1$ cubes. If the large cube is painted on all sides, how many of the small cubes are painted on more than one side? **20**

$\boxed{\text{A}}$ 4 $\boxed{\text{B}}$ 18 $\boxed{\text{C}}$ 20 $\boxed{\text{D}}$ 22 $\boxed{\text{E}}$ 26

6 Eric and his cat leave for a walk. They both walk at constant rates, but Eric is twice as fast as the cat. Eric walks for three miles, then turns around and meets the cat who has been following along the same path. After they meet, the cat follows Eric home. How many miles did the cat walk? **4mi**

A 2   B 3   C 3.6   D 4   E 4.8

7 Given 5 points: $A = (3, 5)$, $B = (0, -5)$, $C = (2, -8)$, $D = (10, 0)$, $E = (6, 5)$. Every minute all the points move by doubling their x-coordinates and halving their y-coordinates. [For example, after one minute, the point A moves to $(6, 2.5)$.] They repeat this every minute for 2,001 years. At that time, which two points are closest? **A,C**

A A, B B B, C C B, E D A, E E A, C

8 Solve for x. $\boxed{\mathbf{64}}$

$$\log_2\left(\log_3(\log_4 x)\right) = 0.$$

$\boxed{\text{A}}$ 24 $\boxed{\text{B}}$ 16 $\boxed{\text{C}}$ 64 $\boxed{\text{D}}$ 81 $\boxed{\text{E}}$ 4096

9 The sum of the first three terms in an increasing arithmetic progression is 12, and the product of the first three terms is -80. Find the third term. $\boxed{\mathbf{10}}$

$\boxed{\text{A}}$ 4 $\boxed{\text{B}}$ 5 $\boxed{\text{C}}$ $\sqrt[3]{80}$ $\boxed{\text{D}}$ 10 $\boxed{\text{E}}$ $4\sqrt{5}$

10 In how many ways can you write 13 distinct letters of the alphabet in alphabetical order without writing any two consecutive letters? (For example, your 13-letter string could not begin with BDE. . . , because D and E are consecutive.) **14**

A 14   B 25   C 26   D 156   E 325

11 The product

$$(1-x+x^2)(1-x^3+x^6)(1-x^9+x^{18})(1-x^{27}+x^{54})\cdots(1-x^{3^{10}}+x^{2\cdot 3^{10}})$$ $\boxed{(3^{11}-1)/2}$

is multiplied out and all like terms collected. How many coefficients are negative?

A 3^{10} B $\dfrac{3^{11}-1}{2}$ C $\dfrac{3^{11}+1}{2}$

D $2\cdot 3^{10}$ E $2\cdot 3^{10}+1$

12 There are 648 three-digit numbers which have distinct digits. Find the average value of these 648 numbers. $\boxed{548\tfrac{8}{9}}$

$\boxed{\text{A}}$ $445\frac{5}{9}$ $\boxed{\text{B}}$ $544\frac{4}{9}$ $\boxed{\text{C}}$ $548\frac{8}{9}$ $\boxed{\text{D}}$ 555 $\boxed{\text{E}}$ $555\frac{2}{3}$

13 Let $ABCD$ be a square with side length 1 unit. Circles with radii 1 are drawn centered at A, B, C, D. What is the area of the intersection of the four circles? $\boxed{\pi/3-\sqrt{3}+1}$

$\boxed{\text{A}}$ $\dfrac{\pi}{3}-\sqrt{3}+1$ $\boxed{\text{B}}$ $\dfrac{\pi\sqrt{3}}{12}$ $\boxed{\text{C}}$ $\dfrac{\pi\sqrt{3}-\sqrt{2}}{12}$

$\boxed{\text{D}}$ $\dfrac{\pi\sqrt{3}+\sqrt{2}}{12}$ $\boxed{\text{E}}$ $\dfrac{\pi}{4}-\dfrac{\sqrt{3}}{2}$

14 Find the sum of the squares of the solutions to the equation $x^3 - 5x^2 + 2x - 11 = 0$. **29**

A 0 B 14 C 21 D 29 E 125

15 Let P be the plane $x + 2y + 3z = 14$, and let **(-39, 20, 4)**

$$A = (100, -50, 5),\ B = (0, 40, -21),\ C = (10, 30, -17),$$

$$D = (-5, 60, -33),\ E = (-39, 20, 4).$$

All but one of these five points lie on the same side of P. Which one?

A A B B C C D D E E

16 Let ABC be a triangle. Length $AB = 7$, and the other two lengths are integers. Point D lies on AC between A and C, such that $AD = 3$ and $BD = 5$. What is the largest possible value for BC? **$\boxed{19}$**

$\boxed{\text{A}}$ 7 $\boxed{\text{B}}$ 19 $\boxed{\text{C}}$ 23 $\boxed{\text{D}}$ 31 $\boxed{\text{E}}$ 43

17 Let S be the set of all positive integers that are relatively prime to 2001. Let $\theta = 2\pi/2001$. Compute **$\boxed{-1}$**

$$\sum_{a \in S} \cos a\theta.$$

$\boxed{\text{A}}$ -1 $\boxed{\text{B}}$ 0 $\boxed{\text{C}}$ 1 $\boxed{\text{D}}$ $\sqrt{2001} - 1$ $\boxed{\text{E}}$ $\sqrt{2001} + 1$

18 Here are the first few rows of Pascal's Triangle, $\boxed{\mathbf{1}}$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1,
\end{array}
$$

where the elements of each row are the sums of pairs of adjacent elements of the prior row. For example, $10 = 4 + 6$.

A number from the first 10^{100} rows of Pascal's Triangle is selected at random. Let X be the probability that this number is divisible by 2001. Which of the following numbers is closest to X?

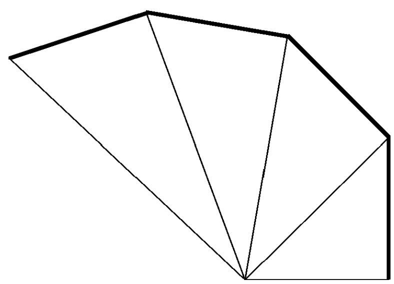
$\boxed{\text{A}}$ 0 $\quad$ $\boxed{\text{B}}$ $\dfrac{1}{2001}$ $\quad$ $\boxed{\text{C}}$ $\dfrac{1}{29} + \dfrac{1}{23} + \dfrac{1}{3}$

$\boxed{\text{D}}$ $\dfrac{28 \cdot 22 \cdot 2}{29 \cdot 23 \cdot 3}$ $\quad$ $\boxed{\text{E}}$ 1

19 Construct a sequence of right triangles as follows. Triangle ABX_1 is an isosceles right triangle with legs of length 1 and hypotenuse AX_1. Triangle AX_1X_2 is constructed by drawing leg X_2X_1 of length 1 perpendicular to AX_1. Likewise, triangle AX_2X_3 is constructed by drawing leg X_3X_2 of length 1 perpendicular to AX_2, etc. The path $BX_1X_2X_3X_4\ldots$ spirals counterclockwise around A. (The figure below shows the first 4 triangles; the path $BX_1X_2X_3X_4$ is darkened.) How many complete revolutions are made by the path **9**

$$BX_1X_2X_3X_4\cdots X_{1000}?$$

A 9 B 10 C 17 D 31 E 32



20 How many terms in the 2002nd row of Pascal's Triangle (the one that begins with 1, 2001 . . .) have a remainder of 1 when divided by 3? (See problem 18 for the definition of Pascal's Triangle.) $\boxed{\textbf{28}}$

$\boxed{\text{A}}$ 23 $\boxed{\text{B}}$ 26 $\boxed{\text{C}}$ 28 $\boxed{\text{D}}$ 29 $\boxed{\text{E}}$ 667

Answers

1. 12π
2. 5/6
3. 3
4. 6
5. 20
6. 4mi
7. A,C
8. 64
9. 10
10. 14
11. $(3^{11}-1)/2$
12. $548\frac{8}{9}$
13. $\pi/3-\sqrt{3}+1$
14. 29
15. (-39, 20, 4)
16. 19
17. -1
18. 1
19. 9
20. 28