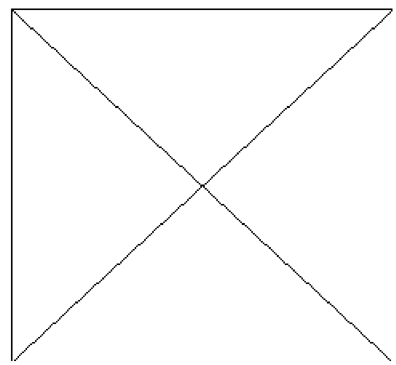
Test of Ingenuity



9TH BAY AREA MATH MEET
UNIVERSITY OF SAN FRANCISCO
April 27, 2002

Directions: Please do not open the test booklet until the proctor begins the examination. This is a multiple-choice, 15-question exam. (In past years, this exam had 20 questions.) You will have one hour to work on the problems. You get one point for a correct answer, zero points for no answer, and $-1/4$ points for an incorrect answer. Because of this penalty for guessing, if you are not sure of the correct answer to a question, it is best not to answer the question.

The questions are arranged in roughly increasing order of difficulty. The last 5 questions are particularly hard and will be used to break ties. Unless you are extremely ambitious, do not attempt all of the problems! Answering just half of them correctly is a very fine achievement! Read the questions first to see which problems are best for you.

Good luck, and have fun!

1 Find the area of a square whose perimeter is 12.

A 6   B 9   C 12   D 18   E 36

2 What is the rightmost (unit's) digit of 7^{2002}?

A 1   B 3   C 5   D 7   E 9

3 Let $A = 1 - x$ and let $B = (1 + x)(1 + x^2)(1 + x^4)$. If $x = 0.1$, compute AB.

A 0.999999   B 0.99999999   C 1

D 1.00000001   E 1.000001

4 Let $d(n)$ denote the number of positive divisors of n, including 1 and n. For example, $d(12) = 6$, since 12 has 6 divisors; 1, 2, 3, 4, 6, 12. Compute $d(60^3)$.

A 112   B 120   C 216   D 1200   E 1728

5 Let ABC be a right triangle with right angle at B, with $AB = 3$ and $BC = 4$. Let D be a point on AC such that BD is perpendicular to AC. Find the area of triangle ABD.

A $9/4$ B $5/2$ C $25/12$ D $54/25$ E $27/16$

6 One percent of the population suffers from a certain disease. There is a blood test for this disease, and it is 99% accurate, in other words, the probability that it gives the correct answer is 0.99, regardless of whether the person is sick or healthy. A person takes the blood test, and the result says that she has the disease. What is the probability that she actually has the disease?

A 0.99%   B 25%   C 50%   D 75%   E 98%

7 Anna, Betül, and Carlos play a game. The positive value x is randomly selected. Then Anna computes the value $100x^2$, Betül computes x^3 and Carlos computes 1.05^x. Then the three people are ranked in order of their number. How many different rankings are possible? Include ties in your count.

A 3 B 6 C 9 D 11 E 12

8 Carl is exactly 6 feet tall. His eyes are exactly 12 inches below the top of his head. He wishes to mount a mirror on the wall so that when he stands 8 feet away, he will see himself from head to toe. What is the minimal height of this mirror in feet? (Assume, of course, that Carl stands straight, perpendicular to the floor, and that the wall is perpendicular to the floor.)

A 3 B $\sqrt{40}$ C $\sqrt{48}$ D 5 E $5\sqrt{2}/2$

9 Find the sum of all values of a for which the quantity $10x + ay$ is maximized by infinitely many points (x, y) satisfying the simultaneous inequalities

$$\begin{aligned} 2x + 3y &\leq 2002, \\ 3x + 2y &\leq 2002, \\ x &\geq 0, \\ y &\geq 0. \end{aligned}$$

A 15   B 48/3   C 65/3   D 79/3   E 82/3

10 Compute $\sin^2 0^\circ + \sin^2 1^\circ + \sin^2 2^\circ + \cdots + \sin^2 90^\circ$.

A 45   B 45.5   C $45\sqrt{2}$ D $91\sqrt{2}/2$ E $45\sqrt{3}$

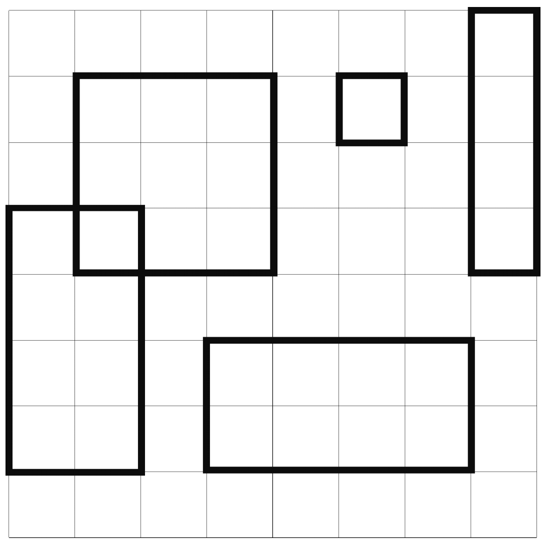
11 Let ABC be an obtuse triangle satisfying

$$AB \cdot BC \cdot CA = 3\sqrt{3} \sin A \sin B \sin C.$$

Let K be the area of this triangle. Then

- [A] K can be any positive value less than $1/2$.
- [B] K can be any positive value less than $\sqrt{3}/2$.
- [C] K can be any positive value less than $3/4$.
- [D] K can be any positive value less than $\sqrt{2}/2$.
- [E] K can be any positive value less than $27\sqrt{3}/32$.

12 How many different rectangles can be formed using the squares of an 8×8 chessboard? For example, there are 64 single-square rectangles, one 8×8 rectangle (the entire chessboard), eight 1×8 rectangles, eight 8×1 rectangles, etc. The picture below shows several more rectangles, of dimension 2×4, 3×3, 1×1, 4×2, and 1×4.



A 1024 B 1156 C 1296 D 2304 E 3136

13 Here are the first few rows of Pascal's Triangle.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

The elements of each row are the sums of pairs of adjacent elements of the prior row. For example, $10 = 4 + 6$. The next row in the triangle will be

$$1, 6, 15, 20, 15, 6, 1.$$

How many elements of the first 64 rows of Pascal's Triangle (the 64th row begins with 1, 63, ...) are not divisibile by 4?

$\boxed{\text{A}}$ 1024 $\boxed{\text{B}}$ 1134 $\boxed{\text{C}}$ 1256 $\boxed{\text{D}}$ 1560 $\boxed{\text{E}}$ 1612

14 Let σ be a **permutation** of the set $\{1, 2, 3, \ldots, n\}$. In other words,

$$\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$$

is just a rearrangement of the numbers from 1 to n, with no duplications or omissions. A permutation is called **cubic** if 3 repeats of this permuation restores the set to its original order. For example, if $n = 7$, the permutation σ defined by

$$\sigma(1) = 3, \sigma(2) = 2, \sigma(3) = 7, \sigma(4) = 4, \sigma(5) = 5, \sigma(6) = 6, \sigma(7) = 1$$

is cubic. Note also that the "identity" permution (the permutation that leaves the set unchanged) is always cubic.

Let x be the number of cubic permutations of the set $\{1, 2, 3, \ldots, 14\}$. Find the last (rightmost) digit of x.

[A] 1 [B] 3 [C] 5 [D] 7 [E] 9

15 Each of the following are products of two primes. Only one of these products can be written as the sum of the cubes of two positive integers. Which one?

- (A) 104729×8512481779
- (B) 104729×8242254443
- (C) 104761×8242254443
- (D) $104761 \times 11401596337$
- (E) $104729 \times 11401596337$