Test of Ingenuity

Bay Area Math Meet
University of San Francisco
April 30, 1994

Directions: Please do not open the test booklet until the proctor begins the examination. This is a multiple-choice, 20-question exam. You will have one hour to work on the problems. There is a penalty for guessing, so if you are not sure of the correct answer to a question, it is best to not answer the question.

The first few questions of this exam are designed to be pretty easy, the next few somewhat difficult, and the last six or so very difficult and rather tricky. In the event of a tie, these last six questions will be used as tiebreakers. Unless you are extremely ambitious, do not attempt all of the problems. Read the questions first to see which problems are best for you. Good luck, and have fun!

1 Find the circumference of a circle with area 81π.

A 9π B 18π C $9\pi^2$ D $18\pi^2$ E 81π

2 For non-zero real numbers, let $f(x) = x^{-1}$. Compute

$$f(f(f(\cdots(f(2))\cdots))),$$

where the above expression contains 20 "f"s.

A $\frac{1}{2^{20}}$ B $\frac{1}{2}$ C 2 D 2^{10} E 2^{20}

3 Mary and Jane start at the same point, and run around a track in the same direction. It takes Mary 2 minutes to complete each lap, and Jane 2 minutes and 15 seconds to complete a lap. After 1 hour, how many times has Mary passed Jane (not including when they had just started running).

A 2 B 3 C 4 D 15 E 240

4 In a circle of radius 1, a chord of length $\sqrt{2}$ is drawn, dividing the interior of the circle into two regions. Find the area of the smaller region.

A $\frac{\pi - 1}{2}$ B $\frac{\pi - 1}{4}$ C $\frac{2\pi - 1}{4}$ D $\frac{\pi - 2}{4}$ E $\frac{\pi\sqrt{2} - 1}{2}$

5 Find the probability that an element picked at random from the set $\{1, 2, 3, \ldots, 1001\}$ is divisible by 7 or 11 (or both).

[A] $\frac{1}{13}$ [B] $\frac{1}{11}$ [C] $\frac{1}{77}$ [D] $\frac{17}{77}$ [E] $\frac{18}{77}$

6 There are 100 people in a room. 95 of them speak Russian, 80 speak Mandarin, and 90 speak Spanish. At least how many speak all 3 languages?

[A] 30 [B] 35 [C] 65 [D] 75 [E] 80

7 A circle is inscribed in a square, and then a square is inscribed in this circle. Find the ratio of the area of the big square to the little square.

[A] $\sqrt{2} : 1$ [B] $\sqrt{3} : 1$ [C] 2:1 [D] $\sqrt{5} : 1$ [E] 3:1

8 Given 3 circles of radius 1 that are all tangent to each other, find the area of the region enclosed between the 3 circles.

[A] $\frac{\sqrt{3} - \pi}{2}$ [B] $\frac{2\sqrt{3} - \pi}{2}$ [C] $\frac{\sqrt{3} - \pi}{3}$ [D] $\frac{3\sqrt{3} - \pi}{3}$

[E] $\frac{\sqrt{3} - 2\pi}{3}$

9 Simplify:

$$\log_2\left(\frac{2^{1+\log_8 27}}{\log_4 8^3 - \log_4 8}\right).$$

[A] -1 [B] 0 [C] 1 [D] 2 [E] 3

10 Find the number of zeros at the end of the base-10 representation of 150! (Recall that for positive integers n, $n! = 1 \cdot 2 \cdots (n-1) \cdot n$.)

[A] 15 [B] 30 [C] 31 [D] 36 [E] 37

11 For which integer n is $1/n$ closest to

$$\sqrt{1{,}000{,}000} - \sqrt{999{,}999}?$$

[A] 1,000 [B] 1,999 [C] 2,000 [D] 2,001 [E] 500,000

12 Compute the infinite series

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \frac{5}{2^5} + \cdots.$$

[A] 1 [B] $\frac{4}{3}$ [C] $\frac{3}{2}$ [D] 2 [E] $\frac{9}{4}$

13 Assume that the earth is a perfect sphere with radius 4,000 miles. Suppose you start at latitude 30 degrees north of the equator (in other words, if your position is P, the center of the earth is O, and the North Pole is N, then the measure of angle PON is 60°). If you travel east until you return to your original position, then how many miles will you travel?

A 2000π B 4000π C $4000\sqrt{2}\pi$ D $4000\sqrt{3}\pi$ E 8000π

14 If $a+b+c=0$ and $a^3+b^3+c^3=27$, find abc.

A -3 B 0 C 3 D $3\sqrt[3]{9}$ E 9

15 Find the length of the shortest path from the point (3,5) to the point (8,2) that touches the x-axis and also touches the y-axis.

A $\sqrt{34}$ B $\sqrt{74}$ C 13 D $2\sqrt{34}$ E $\sqrt{170}$

16 A two-inch elastic band is fastened to the wall at one end, and there's a bug at the other end. Every minute (beginning at time 0), the band is instantaneously and uniformly stretched by 1 inch, and then the bug walks 1 inch towards the fastened end. The bug will reach the wall

A in under 3 minutes.

B in under 6 minutes, but more than 3 minutes.

C in under 10 minutes, but more than 6 minutes.

D in under an hour, but more than 10 minutes.

E never.

17 If John is twice as old as Mary was when Bill was three times as old as Lisa was when she was 17 years younger than John is now, and Mary is 6 years older than Bill, then how old is John?

A 18 B 20 C 23 D 42

E There is not enough information to answer this question.

18 Let $a_0 = 1{,}000{,}000$. For $n \geq 1$, define

$$a_n = \frac{a_{n-1}}{4} + 48.$$

Then a_{100} is closest to

A 48 B 60 C 64 D 72 E 96

19 Bottle A contains a quart of milk and bottle B contains a quart of black coffee. Pour a small amount from B into A, mix well, and then pour back from A into B until both bottles again each contain a quart of liquid. Then

A The fraction of coffee in A is equal to the fraction of milk in B.

B The fraction of coffee in A is bigger than the fraction of milk in B.

C The fraction of coffee in A is less than the fraction of milk in B.

D The answer varies; it depends on how well you do the mixing.

E The answer varies; it depends on how much you pour.

20 For a deck containing an even number of cards, define a "perfect shuffle" as follows: divide the deck into two equal halves, the top half and the bottom half, then interweave the cards one-by-one between the two halves starting with the top card of the bottom half, then the top card of the top half, etc. For example, if the deck has 6 cards, labeled "123456" from top to bottom, after a perfect shuffle the order of the cards will be "415263". Determine the minimum (positive) number of perfect shuffles needed to restore a 94-card deck to its original order.

A 36 B 47 C 54 D 93 E 94

Answers

1 B

2 C

3 B

4 D

5 D

6 C

7 C

8 B

9 C

10 E

11 C

12 D

13 D

14 E

15 E

16 B

17 A

18 C

19 A

20 A