Test of Ingenuity

Bay Area Math Meet
University of San Francisco
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Directions: Please do not open the test booklet until the proctor begins the examination. This is a multiple-choice, 20-question exam. You will have one hour to work on the problems. You get one point for a correct answer, zero points for no answer, and $-1/4$ points for an incorrect answer. Because of this penalty for guessing, if you are not sure of the correct answer to a question, it is best not to answer the question.

The questions are arranged in roughly increasing order of difficulty. The last 6 questions are particularly hard and in the event of a tie score, the first prize will go to the student with the highest score on these six problems. Unless you are extremely ambitious, do not attempt all of the problems! Answering just half of them correctly is a very fine achievement! Read the questions first to see which problems are best for you.

Good luck, and have fun!

1 How many different 4-digit integers are there for which one digit is 5, one digit is 6, and two digits are 7's?

[A] 4 [B] 6 [C] 12 [D] 24 [E] 81

2 Let $p(x) = x^2 + 1$. Then $p(x^2 + 1) =$

[A] $x^2 + 2$ [B] $x^2 + 2x + 2$ [C] $x^4 + 2x^2 + 1$

[D] $x^4 + x^2 + 1$ [E] $x^4 + 2x^2 + 2$

3 In order to compute the area of a square, you measure the length of a side. However, your measurement might have been off by up to 10%. What's the most your area calculation might be off?

[A] 10% [B] 19% [C] 20% [D] 21% [E] 100%

4 A square is inscribed in a semicircle of radius r. Find the area of the square.

[A] $r^2/5$ [B] $r^2/2$ [C] $2r^2/5$ [D] $2r^2/3$ [E] $4r^2/5$

5 A certain video game has a rectangular screen that is 12 inches wide and 9 inches high. You start at the lower right corner of the screen and move up and to the left at a 45 degree angle with the vertical. Any time you go off the left side of the screen, you reappear on the right side at the same height, and similarly, when you go off the top of the screen, you reappear on the bottom, at the same horizontal position. How far must you travel before you return to your starting position?

A $18\sqrt{2}$ B 36 C $36\sqrt{2}$ D $36\sqrt{3}$

E You will never return to your starting position.

6 Suppose $\log_4(\log_3(\log_2 x)) = 5$. Find $\log_2(\log_3(\log_2 x))$.

A $\sqrt{5}$ B 5 C $5\sqrt{5}$ D 10 E 50

7 For how many positive integers n is it true that the sum of the factorials of the first n positive integers is a perfect square? (Recall that the factorial of the positive integer k is defined to be the product of the first k positive integers; for example, the factorial of 4 equals $1 \cdot 2 \cdot 3 \cdot 4 = 24$.)

A 1 B 2 C 3 D 4 E more than 4

8 A dog is walking around on the xy-plane. There is a wall along the positive x-axis, of infinitesimal thickness. The dog is on a leash of length 2 whose end is attached to one side of the wall at the point $(1, 0)$. If the dog is unable to pass through the wall, find the total area which the dog is able to reach.

A $3\pi/2$ B 2π C $5\pi/2$ D 3π E $7\pi/2$

9 Find the smallest positive integer n such that $n!$ is a multiple of $(15!)^2$. (Recall that $n!$ means the factorial of n; see problem 6 above.)

A 22 B 26 C 28 D 30 E 45

10 Given two polyhedra (i.e., solids with polygonal faces), all of whose edges have length 1: a pyramid with a square base, and a tetrahedron (a tetrahedron is composed of four trianglular faces). Suppose we glue the two polyhedra together along a triangular face (so that the attached faces exactly overlap). How many faces does the new solid have?

A 5 B 6 C 7 D 8 E 9

11 Suppose that we decompose the set of positive integers into 3 disjoint arithmetic progressions. Then the common differences of the progressions are (not necessarily in order)

[A] 3, 3, 3 or 2, 3, 6, or 2, 4, 4 [B] 3, 3, 3 or 2, 4, 4

[C] 3, 3, 3 [D] 3, 3, 3 or 2, 3, 6

[E] there are infinitely many possibilities

12 $ABCD$ is a square of side length 1. $EFGH$ is a square that has one vertex on each side of $ABCD$. If the edges of $EFGH$ make an angle θ with the edges of $ABCD$, then the area of $EFGH$ is

[A] $\dfrac{1}{\sin\theta + \cos\theta}$ [B] $\dfrac{1}{1 + \sin 2\theta}$ [C] $\dfrac{1}{\tan\theta + \cot\theta}$

[D] $\dfrac{1}{2}$ [E] $\dfrac{1}{2\cos^2\theta}$

13 Find the area of a regular octagon which circumscribes a circle with radius 1.

[A] $4\sqrt{2-\sqrt{2}}$ [B] $8-4\sqrt{2}$ [C] $4\sqrt{2}-2$

[D] $2\sqrt{2}$ [E] None of the above.

14 In the nation of Klopstockia, every province borders exactly 3 other provinces. It is possible that the number of provinces in Klopstockia is exactly

A 27 B 31 C 32 D 33 E any of these.

15 How many "Friday the 13ths" are possible in a normal 365-day year? (Recall that April, June, September and November each have 30 days, February has 28 days, and all the other months have 31 days.)

A Zero, one, or two. B Only one. C One or two.

D One or three. E One, two, or three.

16 A "perfect power" is a number of the form m^n, where m and n are positive integers greater than 1. How many perfect powers are less than or equal to 2^{12}?

A 72 B 81 C 85 D 87 E 97

17 Suppose it is not true that exactly two of the following 3 statements are true. Also statement 1 is false or it is not true that statements 2 and 3 are both true or both false.

1. Statement 2 is false and my dog is not named “Euclid.”
2. My dog is white and is named “Euclid.”
3. Statement 1 is false and my dog is white.

Then we can conclude that my dog is

[A] white and named “Euclid.”

[B] named “Euclid,” but not white.

[C] white, but not named “Euclid.”

[D] neither white nor named “Euclid.”

[E] none of the above.

18 The tetrakaidecahedron is a polyhedron composed of 14 faces: 6 squares and 8 regular hexagons. At each vertex, two hexagons and one square meet. Let θ be the angle betwen one of the hexagonal faces and the adjacent square face. Find $\cos\theta$.

[A] $-\frac{1}{\sqrt{3}}$ [B] $\frac{1}{\sqrt{3}}$ [C] $-\frac{1}{\sqrt{5}}$ [D] $-\frac{\sqrt{3}}{4}$ [E] $-\frac{1}{3}$

19 You start at the point $(1, 0)$, and each minute you either move a distance 1 to the left or a distance 1 to the right. Each minute, the probability is $1/3$ that you move to the left, and $2/3$ that you move to the right. The probability that you ever get to the origin is

[A] $3/7$ [B] $1/2$ [C] $2/3$ [D] $3/4$ [E] 1

20 $\sin 10^\circ \sin 20^\circ \sin 30^\circ \sin 40^\circ \sin 50^\circ \cdots \sin 170^\circ =$

[A] $\frac{1}{2^9}$ [B] $\frac{9}{2^{10}}$ [C] $\frac{3\sqrt{2}}{2^{12}}$ [D] $\frac{9}{2^{16}}$ [E] $\frac{3\sqrt{2}}{2^{16}}$

Answers

1 C

2 E

3 D

4 E

5 C

6 D

7 B

8 C

9 C

10 A

11 B

12 B

13 E

14 C

15 E

16 B

17 B

18 A

19 B

20 D