Key to Test of Ingenuity **Bay Area Math Meet** **April 22, 1995**

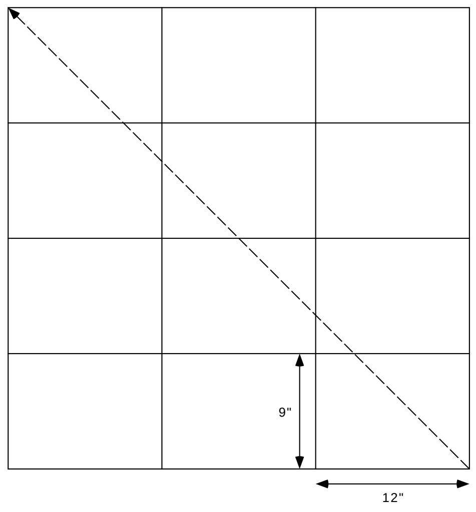
1 The answer is 12. You can see this by just listing them, or using the combinatorial formulas: there are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways of permuting the 4 digits, but 2 of them are the same, so we have overcounted by $2! = 2$. Hence the answer is $4!/2!$.

2 The answer is $x^4 + 2x^2 + 2$. We have $p(x^2 + 1) = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 1 + 1$.

3 The answer is 21%. The worst that could happen is if we overestimated both sides by 10%. Then we would overestimate the area by a factor of $1.1^2 = 1.21$, i.e. we are 21% off.

4 The answer is $4r^2/5$. Let $2x$ denote the length of a side of the square. Notice that the center of the circle coincides with the midpoint of a side of the square. By the Pythagorean theorem, $x^2 + (2x)^2 = r^2$, or $5x^2 = r^2$. Now multiply both sides by 4/5 to get the area of the square.

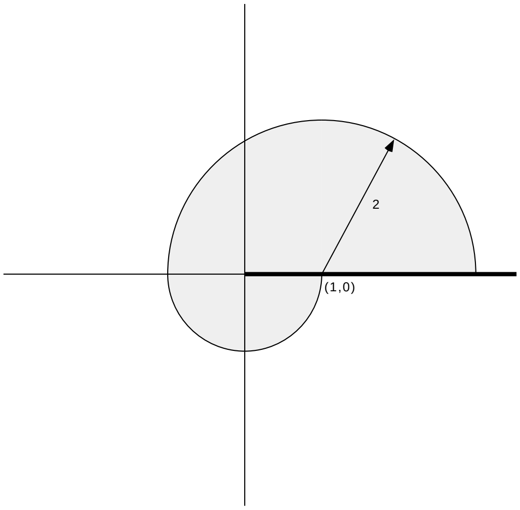
5 The answer is $36\sqrt{2}$. A nice way to do this is to imagine tiling the entire plane with copies of this screen. Then start at the lower-right hand corner, and draw a line which goes up to the left at a 45 degree angle. We need to see when the line first hits a corner of one of the tiled screens. Now an easy application of the Pythagorean theorem will determine the length of the path!



6 The answer is 10. Since $\log_4(\log_3(\log_2 x)) = 5$, then $\log_3(\log_2 x) = 4^5$. Hence $\log_2(\log_3(\log_2 x)) = \log_2 4^5 = \log_2 2^{10} = 10$.

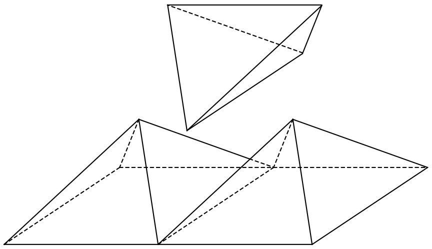
7 The answer is 2. Notice that $1! = 1 = 1^2$, so one answer is $n = 1$. Likewise, experimentation shows that $n = 2$ doesn't work, but $n = 3$ does, since $1! + 2! + 3! = 1 + 2 + 6 = 9$, a perfect square. When $n = 4$ the sum is $1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33$, which is not a perfect square. Now notice that for all $k \geq 5$, the last digit of $k!$ is zero. This means that for all $k \geq 5$, the sum $1! + 2! + 3! + \cdots + k!$ will end with the digit 3. But no perfect square ends with a 3 (why?); hence the only solutions are $n + 1, 3$.

8 The answer is $5\pi/2$. Just look at the picture!



9 The answer is 28. Note that 13^2 is the highest power of 13 that divides $(15!)^2$. (In fancier notation, this is written as $13^2 \| (15!)^2$.) In order for $n!$ to be a multiple of $(15!)^2$, we need to find two 13's in the factorization of $n!$. Since 13 is a prime, this forces n to be at least as big as 26. Likewise, note that $7^4 \| (15!)^2$ (because 15! contains both 7 and 14 as factors). However, $7^3 \| 26!$, since the only factors of 26! which contribute 7s are 7, 14, and 21. To get that extra exponent, we need to go up to 28!.

10 The answer is 5. The original pyramid has 5 faces. Attaching the tetradehron adds *no* new faces! The reason: when the tetrahedron is glued to the pyramid, two of the faces of the tetrahedron are flush with faces of the pyramid! Of the two other faces of the tetrahedron, one is glued to a face of the pyramid (and thus both of these faces now lie inside the new solid), and the other becomes one of the faces of the new solid. The big question is, how to show that the faces are flush? One way is to do rather difficult 3-dimensional vector calculations, with a fair amount of trig. This is a good exercise, and well worth doing. The ingenious way to see it it in a flash is to imagine two pyramids sitting next to each other as shown. It is easy to see that the "gap" between them is exactly the size and shape of a regular tetrahedron. Hence, when you place a regular tetrahedron into this gap, you get a nice prism. In other words, two of the faces of the tetrahedron will be flush with two faces of the pyramid.



11 The answer is 3, 3, 3 or 2, 4, 4. An arithmetic progression of positive integers with common difference k will include $1/k$ of all positive integers. (This statement is a little imprecise, but should make sense. For example, the even integers encompass 50%, or 1/2 of all integers. You might say, "This makes intuitive sense, but can you be more rigorous? We are dealing with infinite sets, you know." A more rigorous formulation is the following: For any subset S of the positive integers $\mathbf{N}$, we define the *density* of S, denoted by $d(S)$, to be the limit

$$\lim_{n\to\infty} \frac{|S \cap \{1, 2, 3, \ldots, n\}|}{n},$$

where $|A|$ denotes the number of elements in the set A. This limit may not always exist. Check for yourself that this definition gives $d(\text{even positive integers}) = 1/2$ and $d(\text{any arithmetic progression with common difference } k) = 1/k$.)

Thus, if $\mathbf{N}$ is decomposed into three disjoint arithmetic progressions with common differences of a, b, c, we must have $1/a + 1/b + 1/c = 1$. There are only finitely many positive integer solutions (a, b, c) to this equation – they can't all be greater than 3! – and a little trial-and-error yields the complete list of $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$. The first two of these lead to the the decompositions

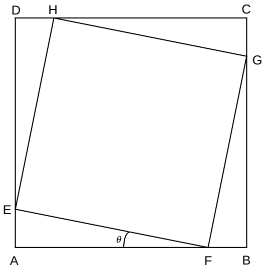
$$\mathbf{N} = \{1, 4, 7, \ldots\} \cup \{2, 5, 8, \ldots\} \cup \{3, 6, 9, \ldots\}$$

and

$$\mathbf{N} = \{1, 3, 5, , \ldots\} \cup \{2, 6, 10, \ldots\} \cup \{4, 8, 12, \ldots\},$$

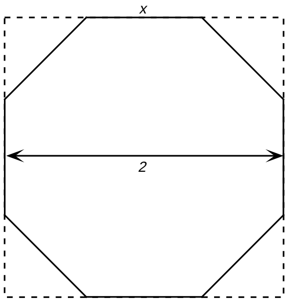
respectively. The third possiblity, (2, 3, 6) cannot work, because all of the elements of an arithmetic progression with common difference 2 have the same *parity* (i.e. all are even, or else all are odd), while the elements of an arithmetic progression with common difference 3 have alternating parity (why?). Hence these two arithmetic progressions cannot be disjoint.

12 The answer is $1/(1+\sin 2\theta)$. Let $EF = x$. Then $AE = x\sin\theta$ and $DE = AF = x\cos\theta$. Consequently, $x(\sin\theta + \cos\theta) = 1$. Now square both sides and use trig identities to find x^2.



13 The answer is $8\sqrt{2} - 8$. One method uses nasty trig, including the half angle formulas. This is worth doing. But we will present another method, which is not necessarily easier, but may be interesting. Let the length of one side of the octagon be x. Extend the sides of the octagon to form a square as shown. The area of the octagon is equal to the radius of the inscribed circle times half the perimeter, or $4x$. (If you haven't seen this easy area formula before, verify it – it can come in very handy!). Since the diameter of the inscribed circle is 2, the side of the square is 2, and since the dotted-line corners of the square are isosceles right triangles, we have $\sqrt{2}x/2 + x + \sqrt{2}x/2 = 2$. Our answer is then

$$4\left(\frac{2}{1+\sqrt{2}}\right).$$



14 The answer is 32. Imagine if we asked the governor of each province, "How many provinces does your province border?" and then added up all of the answers. The result must be even, because whenever the governor of province X counts province Y as a neighbor, the governor of province Y counts province X as a neighbor! So there is "doublecounting" going on. This is another example of the well-known "handshake lemma," which states that if a bunch of people shake hands, the sum of the number of handshakes that each person experiences will be exactly twice the number of handshakes that actually ocurred. So if p is the number of provinces, we deduce that $3p$ must be even, which forces p to be even.

15 The answer is 1, 2, or 3. Let us denote the days of the week by the numbers 0,1,2,3,4,5,6, and reduce everything modulo 7, i.e., we will replace all numbers by their remainder upon division by 7. Let x be the day of the week that January 13 falls on. Then February 13th will fall on $x+31 \equiv x+3 \pmod 7$. This is convenient shorthand for saying that February 13th will fall 3 days later in the week than January 13th. Likewise, March 13th will fall on $x+3+28 \equiv x+3 \pmod 7$. Continuing in this manner, we obtain the list

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
x	$x+3$	$x+3$	$x+6$	$x+1$	$x+4$	$x+6$	$x+2$	$x+5$	x	$x+3$	$x+5$

Now it is clear – you can have 1, 2 , or 3 Friday the 13ths in a year, depending on what x was. This list gives much more information than that. For example, if a year contains exactly one Friday the 13th, it must fall in May, June, August, or September. Likewise, if a year has 3 Friday the 13ths, they must happen during February, March, and December. Question: would leap years have changed the answer?

16 The answer is 81. We must avoid overcounting; it is convenient to only look at numbers of the form m^p, where m is greater than 1 and p is a *prime*. The number of perfect squares which are greater than 1 and less than or equal to 2^{12} is $2^6 - 1 = 63$. Likewise, there are $2^4 - 1 = 15$ perfect cubes and (after some calculation), 4 perfect 5th powers, 2 perfect 7th powers, and one perfect 11th power. Note that we need not check larger primes, since we stop at 2^{12}. Also note that this has counted at least all the perfect powers, since every 4th power is also a square, every 6th power is also a cube, etc.

Now we need to remove overcounting. We must subtract from this list those numbers which are simultaneously perfect squares and perfect cubes, etc. If a number is a perfect square and a perfect cube, it is a perfect 6th power, and there are $2^2 - 1 = 3$ of those. Likewise, if a number is a perfect square and a perfect 5th, it is a perfect 10th power, and there is just one. There are no other cases of overcounting (any other product of primes is greater than 12). So the final count is $63 + 15 + 4 + 2 + 1 - 3 - 1$.

17 The answer is: the dog is named "Euclid," but not white. The first two sentences (before the numbered statements) clearly imply that statement 1 must be false and that of the remaining two statements, at most one is true. Consider the color of the dog. If the dog is white, this makes statement 3 true, which forces statement 2 to be false, which forces the dog not to be "Euclid," which makes statement 1 *true*, a contradiction. Hence the dog is not white, making statements 3 and 2 both false. Since statement 2 is false (dog not white) then the falsity of statement 1 implies that the dog is named "Euclid."

18 The answer is $-1/\sqrt{3}$. A key insight is to realize that the tetrakaidecahedron is just a truncated octahedron, i.e., an octahedron with each vertex "cut off" into a square. To understand the following, you will need to draw a picture. Let the vertices of the octahedron be $ABCDEF$, where $ABCD$ is a square and E and F are respectively above and below the plane of the square. Then one square face of the tetrakaidecahedron will be formed by the intersection of this octahedron with a plane parallel to $ABCD$. Now let G and H be the midpoints of AD and BC respectively. Then the rhombus $EDFH$ is perpendicular to to this square face. Let the measure of angle EGH be β. It is easy to see that $\theta + \beta = \pi$ and that $\cos\beta = 1/\sqrt{3}$. Thus $\cos\theta = -1/\sqrt{3}$.

Another method is to use coordinates as follows. It is messy, but worth studying: At each vertex of the solid, a square and two regular hexagons meet. Assume that the square lies in xyz-space, with its vertices at $A = (0,0,0)$, $B = (1,0,0)$, $C = (0,1,0)$, and $D = (1,1,0)$. Now suppose that $ABRSTU$ are the vertices of any regular hexagon. The diagonal RU has length 2, is parallel to AB, and has the same perpendicular-bisecting-plane as AB. Therefore the x-coordinate of U is $-1/2$. Similarly, if $ADWXYZ$ is any regular hexagon, then Z must have y-coordinate $-1/2$.

Now if $ABCD$ is a square face of the solid, then A is also a vertex of two hexagonal faces. Suppose that one of them is $ABRSTU$, and the other is $ADWXYU$. As shown above, both the x and y coordinates of U are $-1/2$. Since AU has length 1, U must be $(-1/2, -1/2, \pm\sqrt{2}/2)$.

We now calculate the cosine of the angle between $ABCD$ and $ABRSTU$. Since both polygons are parallel to the x-axis, we can first project onto the yz-plane. The square projects to the y-axis, and the hexagon projects to a line through the points $(0,0,0)$ and $(0, -1/2, \pm\sqrt{2}/2)$. Our task reduces to finding the cosine of the angle between these two lines. This is evidently $(-1/2)/(\sqrt{(-1/2)^2 + (\pm\sqrt{2}/2)^2}) = -1/\sqrt{3}$.

19 The answer is 1/2. Let p be the probability that you get to the origin from the starting point of $(0,1)$. The key observation is that p is *also* the probability that you eventually move one unit to the left, starting from any initial position. Let us now compute p. Starting from $(1,0)$ there are two possibilities.

1. With probability $1/3$ we move to $(0,0)$ immediately. In this case, we are done (we have already achieved our goal).
2. With probability $2/3$ we move one unit to the right, to $(2,0)$. To reach $(0,0)$ we need to eventually get back to $(1,0)$. But this has probability p! Then, once we get to $(1,0)$, we need to eventually move to $(0,0)$. Once again, the probability is p. Since the events are independent, we multiply the probabilities, and conclude that the probability is $2p^2/3$ that we eventually reach $(0,0)$ if the very first move was from $(1,0)$ to $(2,0)$.

Adding the two cases, we see that

$$p = \frac{1}{3} + \frac{2}{3}p^2.$$

The two solutions of this quadratic equation are 1 and $1/2$. We reject the first solution, since it disagrees with our intuition, since after all, we tend to "drift" a positive distance to the right each minute, on the average. A rigorous proof that $p \neq 1$ is pretty difficult. Here is a sketch of one which uses the powerful, but sophisticated technique of generating functions: Define $p(x,t)$ to be the probability that we are at $(x,0)$ at time t minutes. Assume that once we get to $(0,0)$, we stay there. Then we need to show that $p(0,t) \leq 1/2$ for all t.

We have $p(1,0) = 1$, and for each t, $p(0,t+1) = p(1,t)/3 + p(0,t)$ and $p(1,t+1) = p(2,t)/3$, and for all $x > 1$, $p(x,t+1) = 2p(x-1,t)/3 + p(x+1,t)/3$. Now define

$$r_t = \sum_{x=0}^{\infty} \frac{p(x,t)}{2^x}.$$

A careful computation shows that $r_t = r_{t+1}$ for all $t \geq 0$. Since it is easy to see that $r_0 = 1/2$, we conclude that $r_t = 1/2$ for all t. This forces $p(0,t) \leq 1/2$, for all t, for otherwise r_t would exceed 1/2 for some t.

20 The answer is $18/2^{17}$. Let us consider a more general problem. A good strategy for attacking problems of this kind is to look at nth roots of unity. If n is a positive integer, the roots of the polynomial $x^n - 1$ are the nth roots of unity $1, \zeta, \zeta^2, \zeta^3, \ldots, \zeta^{n-1}$, where $\zeta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. If we exclude the root 1, the remaining $n-1$ complex numbers are the roots of $(x^n-1)/(x-1)$. In other words,

$$1 + x + x^2 + \cdots + x^{n-1} = (x-\zeta)(x-\zeta^2)(x-\zeta^3)\cdots(x-\zeta^{n-1}).$$

Letting $x = 1$, we get

$$n = (1-\zeta)(1-\zeta^2)(1-\zeta^3)\cdots(1-\zeta^{n-1}).$$

Now take the absolute value of both sides, and use the fact (which can be proven in an ugly way with trig identities or in a pretty way by drawing a picture and looking at vectors) that

$$|1-(\cos\theta+i\sin\theta)| = 2\sin\frac{\theta}{2}.$$

Putting this all together yields

$$n = 2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\cdots\sin\frac{(n-1)\pi}{n}.$$

Finally, let $n = 18$.