Test of Ingenuity

Bay Area Math Meet
University of San Francisco
April 20, 1996

Directions:

- Please do not open the test booklet until the proctor begins the examination.
- *No calculators are allowed.*
- This is a 20-question exam. You will have one hour to work on the problems.
- The first 16 problems are multiple choice and you get one point for a correct answer, zero points for no answer, and $-\frac{1}{4}$ points for an incorrect answer. Because of this penalty for guessing, if you are not sure of the correct answer to a question, it is best not to answer the question.
- The last four problems have answers which are whole numbers between 0 and 999. You will mark these answers on a special space on the answer form as explained by the proctor. You get one point for a correct answer and 0 points for an incorrect or blank answer.
- The questions are arranged in roughly increasing order of difficulty. The last 6 questions are particularly tricky and in the event of a tie score, the first prize will go to the student with the highest score on these six problems.
- Unless you are extremely ambitious, do not attempt all of the problems! Answering just half of them correctly is a superb achievement! Read the questions first to see which problems are best for you.
- Good luck, and have fun!

1 Let $f(x) = x^2 - 1$. Then $f(f(x)) =$

A $2x^2 - 2$ B $x^4 - 1$ C $x^4 - 2x^2 - 1$

D $x^4 - 2x^2$ E $x^4 - 2x^2 + 1$

2 Let ABC be a triangle, and let D be the midpoint of AC. What is the ratio of the area of triangle ABD to the the area of triangle DBC?

A Cannot be determined with the information given.

B $\frac{1}{3}$ C $\frac{1}{2}$ D $\frac{4}{9}$ E $\frac{1}{1}$

3 A survey of people who eat at two different fast food restaurants determines that 72% believe that Bay Area Rapid Food burgers are as tasty or tastier than Church of Burgerology burgers. If x% believe that Church of Burgerology burgers are as tasty or tastier than Bay Area Rapid Food burgers, then

A $x = 28$ B $0 \leq x \leq 28$ C $28 \leq x \leq 44$

D $28 \leq x \leq 72$ E $28 \leq x \leq 100$

4 Suppose we reflect the point (x, y) about the line $x = 10$, then about the line $x = 9$, then about $x = 8$, and then about $x = 7$. The resulting point will be:

[A] $(x-4, y)$ [B] $(x-2, y)$ [C] (x, y)

[D] $(x+2, y)$ [E] $(x+4, y)$

5 In a triangle with sides 3, 4 and 5, what is the distance between the centers of the inscribed circle and circumscribed circle?

[A] $\frac{\sqrt{2}}{2}$ [B] $\frac{\sqrt{3}}{2}$ [C] 1 [D] $\frac{\sqrt{5}}{2}$ [E] $\frac{5}{4}$

6 In triangle ABC, $AB = 3$ and $BC = 4$ and the measure of $\angle ABC = 90^\circ$. A point Q is located on the line segment AC. Let P be a point outside triangle ABC such that the line segment PQ is parallel to AB and $PQ = 1$. The area of triangle APC is

[A] 1 [B] 2 [C] $\sqrt{2}$ [D] $2\sqrt{2}$

[E] not determined from the given information.

7 Find x, if $3^{(27^x)} = 27^{(3^x)}$.

A $\frac{1}{3}$ B $\frac{1}{2}$ C $\frac{1}{\sqrt{3}}$ D 1 E $\sqrt{3}$

8 Let ABC be an equilateral triangle with side length 1. Construct circles with unit radii centered at A, B and C. What is the area of the intersection of these three circles?

A $\frac{\pi - \sqrt{3}}{2}$ B $\frac{2\pi - \sqrt{3}}{2}$ C $\frac{\pi - 2\sqrt{3}}{2}$ D $\frac{\pi + \sqrt{3}}{2}$
E $\frac{4\pi - \sqrt{3}}{4}$

9 A Klopstockia Lottery ticket sells for \$1. When the buyer scratches the surface of the ticket, it will announce the prize. The prizes are \$1, \$10, \$1,000, and a free lottery ticket. The probabilities of these events are respectively $\frac{1}{10}$, $\frac{1}{1{,}000}$, $\frac{1}{1{,}000{,}000}$ and $\frac{1}{5}$. To the nearest penny, the state of Klopstockia makes a net average profit of how much per ticket sold?

A \$0.85 B \$0.86 C \$0.87 D \$0.88 E \$0.89

10 Select points A, B and C randomly on the circumference of a circle. What is the probability that angle ABC is obtuse?

A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{8}$ E $\frac{2}{9}$

11 Let A, B, C and D be points in the plane with coordinates (a, x), (b, y), (c, z) and (d, w), respectively. Suppose that

$$a + b + c + d = x + y + z + w = 0$$

and

$$a^2 + x^2 = b^2 + y^2 = c^2 + z^2 = d^2 + w^2 = 1.$$

Then the measure of $\angle ABC$

A is always less than 90 degrees.

B is always greater than 90 degrees.

C is always equal to 90 degrees.

D can be any value between 45 and 135 degrees.

E can be any value between 0 and 180 degrees.

12 Let C be the circle $x^2 + y^2 = 1$. The line ℓ intersects C at the point $(-1, 0)$ and the point P. Suppose that the slope of ℓ is a rational number m. Then for how many choices of m are both coordinates of P rational numbers?

A 3 B 11 C 12 D all rational m

E none of the above

13 Pat works in San Francisco and lives in Oakland with Chris. Every afternoon, Pat gets on a BART train which arrives at the Rockridge station in Oakland at exactly 5pm. Chris leaves the house before 5 and drives at a constant speed so as to arrive at the station at exactly 5pm to pick up Pat. The route that Chris drives never changes.

One day, Pat decides to leave early, and catches a train which arrives at the Rockridge station at 4pm. Instead of phoning Chris to ask for an earlier pick-up, Pat decides to get a little exercise, and begins walking home along the route that Chris drives, knowing that eventually Chris will intercept Pat, and then will make a U-turn, and they will head home together in the car. This is indeed what happens, and Pat ends up arriving at home 10 minutes earlier than on a normal day. Assuming that Pat's walking speed is constant, and that the U-turn takes no time, and that Chris's driving speed is constant, for how many minutes did Pat walk?

A 55 B 50 C 45 D 30

E cannot be determined without more information

14 Assume that n is a perfect square which is greater then 1,000,000. Let

$$a = 2^{(\sqrt{n})!}, \quad b = 2^{2^n}, \quad c = (2^n)!.$$

List these three quantities from largest to smallest.

A a, b, c B b, c, a C c, b, a

D c, a, b E b, a, c

15 At first, a room is empty. Each minute, either one person enters or two people leave. Which of the numbers below *cannot* be a possible population of the room after exactly 10^{1996} minutes?

A 10^{1995} B 10^{998} C $(10^{994}+1)^2$

D $10^{1001}+10^{899}+10^{90}+10^{6}$ E $10^{1000}+10^{900}+10^{96}$

16 On Joe's 100th birthday, he receives a cake with 100 candles. He wants to cut the cake with straight cuts, so that each piece contains exactly one candle. He arranges the candles so that this can be accomplished with the least number of cuts possible. How many cuts will be needed? Assume that the cake is flat, that the candles are only on the top of the cake, and that the cuts are vertical (perpendicular to the top of the cake).

A 8 B 14 C 15 D 16 E 17

For problems 17–20 below, the answers are integers between 0 and 999. Please fill in your answers on the answer sheet in the spaces provided, which are marked with the letters "ABCDEFGHIJKL" and located to the right of the box labeled "SOCIAL SECURITY NUMBER."

Don't forget to include leading zeros in your answer. For example, if your answer is 42, you will mark "042" on the answer sheet in the appropriate spaces.

17 Let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers whose sum is 13. For each $x > 0$, define $S(x)$ to be the minimum value of the sum

$$\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \cdots + \sqrt{a_n^2 + b_n^2},$$

taken over all sequences $b_1, b_2, \ldots, b_n$ of real numbers whose sum is x. If x and $S(x)$ are both positive integers, then what is x? *(Mark the answer in columns "ABC.")*

18 A spherical, 3-dimensional planet has center at $(0, 0, 0)$ and radius 20. At any point (x, y, z) on the surface of this planet, the temperature is equal to $(x+y)^2+(y-z)^2$ degrees. What is the average temperature of the surface of this planet, rounded to the nearest degree? *(Mark the answer in columns "DEF.")*

19 Consider the line segments joining every possible pair of vertices of a regular heptagon (7-gon) which is inscribed in a circle with radius 1 unit. The product of the lengths of these segments is $a\sqrt{b}$, where a and b are integers and no perfect square divides b. Find $a + b$. *(Mark the answer in columns "GHI.")*

20 Given any sequence of n distinct integers, we compute its "swap number" in the following way: Reading from left to right, whenever we reach a number which is less than the first number in the sequence, we swap its position with the first number in the sequence. We continue in this way until we get to the end of the sequence. The swap number of the sequence is the total number of swaps. For example, the sequence 3, 4, 2, 1 has a swap number of 2, for we swap 3 with 2 to get 2, 4, 3, 1 and then we swap 2 with 1 to get 1, 4, 3, 2.

Let x be the average value of the swap numbers of the $7! = 5040$ different permutations of the integers $1, 2, 3, 4, 5, 6, 7$. What is $100x$, rounded to the nearest integer? *(Mark the answer in columns "JKL.")*

Answers

1. D
2. E
3. E
4. A
5. D
6. B
7. B
8. A
9. B
10. C
11. C
12. D
13. A
14. C
15. E
16. B
17. 084
18. 533
19. 350
20. 159