Key to Test of Ingenuity **Bay Area Math Meet** **April 20, 1996**

COMMENT: The 1996 Test of Ingenuity was the hardest ever. We made it hard because we were embarrassed that Chris Chang of Palo Alto, arguably the best math student in the nation, got perfect scores two years in a row, and decided to make an exam that he would find challenging. Unfortunately, Chris didn't attend BAMM '96. The top score on this exam (just over 15 points) went to Li-Chung Chen, a superb accomplishment! We vow that the 1997 Test of Ingenuity will be somewhat easier.

1 The answer is x^4-2x^2. We have $f(f(x)) = (x^2-1)^2-1 = x^4-2x^2+1-1 = x^4-2x^2$.

2 The answer is $1/1$. Both triangles have the same altitude (dropped from B to line AC, and equal base lengths ($AD = DC$).

3 The answer is $28 \le x \le 100$. Exactly 100-72= 28% do *not* believe that BARF burgers are as tasty or tastier than COB burgers, i.e. 28% believe that COB burgers are strictly tastier than BARF burgers. However, the remaining 72% may or or may not believe that COB burgers are as tasty as BARF burgers, so the total fraction who think that COB burgers are as tasty or tastier than BARF burgers is at least 28% and possibly as large as 100%.

4 The answer is $(x-4, y)$. When (x, y) is reflected about $x = a$, the outcome is $(2a-x, y)$. (Draw a picture.) Thus we have $(x, y) \mapsto (20-x, y) \mapsto (18-(20-x), y) = (-2+x, y) \mapsto (16-(-2+x), y) = (18-x, y) \mapsto (14-(18-x), y) = (x-4, y)$. Alternatively, we can proceed more geometrically. It is easy to check to check (draw a picture!) that reflection about the line $x = a$ followed by reflection about $x = a-1$ is equivalent to translation by 2 units to the left. This process is repeated twice, so $(x, y) \mapsto (x-4, y)$.

5 The answer is $\sqrt{5/4}$. Let the triangle be ABC, where $AB = 3$, $AC = 4$ and $BC = 5$. Since $\angle CAB = 90^\circ$, CB is the diameter of a circle which also goes through A, i.e. the center of the circumscribed circle of triangle ABC is the midpoint of BC. Let us call that point D. Next, let E denote the incenter (center of the inscribed circle). Let the inradius (radius of the inscribed circle) be r. Notice that we can decompose triangle ABC into three triangles: DAB, ECA and ECB. Dropping perpendiculars from E to each side of ABC and computing areas, we have (using the notation $[XYZ]$ for the area of triangle XYZ)

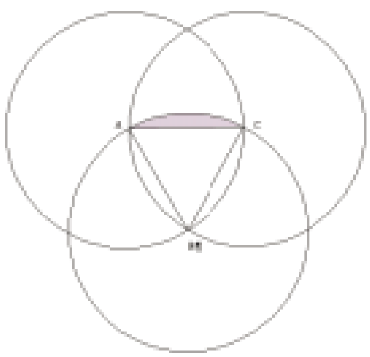
$$6 = [ABC] = [EAB] + [ECA] + [ECB] = \frac{3}{2}r + \frac{4}{2}r + \frac{5}{2}r.$$

Hence $r = 1$. There are many ways to find the distance from E to D; one method is coordinates. If we put A at the origin and $B = (3, 0)$ and $C = (0, 4)$, then $E = (1, 1)$ and $D = (\frac{3}{2}, 2)$. So the distance is $\sqrt{(\frac{3}{2} - 1)^2 + (2 - 1)^2} = \sqrt{\frac{5}{4}}$.

6 The answer is 2. Triangle APC divides into two triangles, QPC and QPA. Each of these triangles has base PQ. The total of the altitudes of the two triangles is equal to $BC = 4$. Hence $[APC] = [QPC] + [QPA] = \frac{1}{2} PQ \cdot 4 = 2$.

7 The answer is $1/2$. Reducing everything to powers of 3, the original equation becomes $3^{(3^{3x})} = 3^{3 \cdot 3^x}$. Hence $3^{3x} = 3 \cdot 3^x$, which implies that $3x - 1 = x$, or $x = 1/2$.

8 The answer is $(\pi - \sqrt{3})/2$.



The area that we seek is $T + 3S$, where $T = \sqrt{3}/4$ is the area of the equilateral triangle ABC and S is the shaded area. Since $S + T$ is a one-sixth section of a full circle, we have $S + T = \pi/6$. So $T + 3S = T + 3(\pi/6 - T) = \pi/2 - 2T$. Now plug in the value of T.

9 The answer is \$0.86. Let x equal the *expected value* of the lottery ticket. This is equal to the sum of the products of each possible prize times the probability of that prize. Hence we have

$$x = 1 \cdot \frac{1}{10} + 10 \cdot \frac{1}{1,000} + 1000 \cdot \frac{1}{1,000,000} + x \cdot \frac{1}{5},$$

where the last term is due to the fact that a lotter ticket has value x. Solving for x, we get $x = \$0.13875$, so the net average profit will be $\$1.00 - x = \$0.86$, to the nearest penny.

10 The answer is $1/4$. Without loss of generality, fix the point A, and draw a diameter which passes through A. This diameter divides the circumference of the circle into two semicircles, call them U and V. The probability that the point B lies in U is clearly 1/2;

likewise the probability that B lies in V is 1/2, etc. The probability that both B and C lie in the same semicircle is 1/2 (since the probability is 1/4 that they both lie in U and 1/4 that they both lie in V. But that is not enough to insure that $\angle ABC$ is obtuse, since the *order* matters: For every configuration where B and C lie in the same semicircle and B lies between A and C there is another configuration where B and C lie in the same semicircle but C lies between A and B. Only in the first case will $\angle ABC$ be obtuse. In other words, we must divide our original probability of 1/2 by 2.

11 The answer is, "always equal to 90 degrees." The first condition is equivalent to saying that the center of mass of the points A, B, C, D is the origin. The second condition says that all four points lie on the circumference of a circle with center at the origin and radius 1. Now pick point A, and draw the diameter through A. Since the center of mass is the origin, i.e., lies on this diameter, the remaining three points must be arranged so that they "balance" with respect to this diameter. This implies that one of the three points lies on the diameter directly opposite from A. Since this principle applies to any two points (given any of the four points, one of the other points lies directly opposite it on a diameter), the four points must make a rectangle!

12 The answer is all rational m. We have $y/(x+1) = m$ and $x^2 + y^2 = 1$. Substituting for y yields the quadratic equation $(m^2+1)x^2+2m^2x+m^2-1 = 0$. Now observe that this factors into $((m^2+1)x+(m^2-1))(x+1) = 0$, so $x = -1$, $(1-m^2)/(1+m^2)$. Thus x is rational, and so therefore y will also be rational. (Alternatively, we could have used the quadratic formula, merely checking that the discriminant $b^2-4ac = 4m^4-4(m^2-1)(m^2+1) = 4$ is a perfect square.

13 The answer is 55 minutes. Looking at things from Chris's point of view, Chris drives for 10 minutes less than normal, which means that the time of pick-up is 5 minutes earlier than normal (since time is gained equally in both directions). Thus Pat was picked up at 4:55.

14 The answer is c, b, a. Clearly $u! > 2^u$ for all integers $u > 2$, which implies that $c > b$ (just let $u = 2^n$). To establish that $b > a$, we need to show that $(\sqrt{n})! < 2^n$. Let $n = m^2$. Then we wish to show that $m! < 2^{m^2}$. This is certainly true because $2^{m^2} = (2^m)^m$ and $2^m > m$ for $m > 2$. So the entire problem really was equivalent to the easy inequalities

$$u < 2^u < u!$$

for $u > 2$.

15 The answer is $10^{1000} + 10^{900} + 10^{96}$. Modulo 3, each minute the population increases by $+1$, since $+1 \equiv -2 \pmod 3$. Since $10^{1996} \equiv 1 \pmod 3$, the population must be congruent to 1 modulo 3 after 10^{1996} minutes. The only choice that is not congruent to 1 modulo 3 is $10^{1000} + 10^{900} + 10^{96}$.

16 The answer is 14. Let $f(n)$ denote the maximum number of areas that you can cut the cake into with n straight cuts. Clearly $f(1) = 2$, $f(2) = 4$, $f(3) = 7$. We conjecture the recurrence $f(n+1) = f(n) + n + 1$. This is true because at best, the $n+1$th cut will intersect all of the other n cuts, producing n new areas. But as the line exits the circle, one more area is produced (draw a picture!). Thus

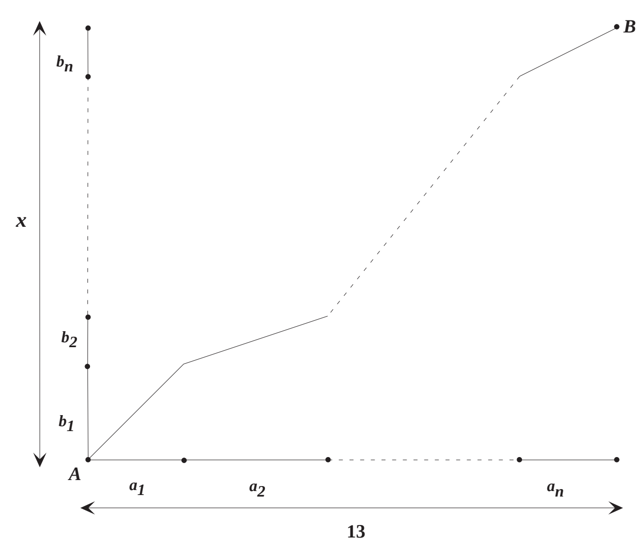
$$f(k) = k + f(k-1) = k + (k-1) + f(k-2) = \cdots = k + (k-1) + (k-2) + \cdots + 2 + 1 + f(1),$$

so $f(k) = \dfrac{k(k+1)}{2} + 1$. Hence $f(13) = 92$ and $f(14) = 106$ so 13 cuts are not enough; 14 are needed.

17 The answer is 84. Using the Pythagorean theorem, interpret each term $\sqrt{a_i^2 + b_i^2}$ as the hypotenuse of a triangle. Then

$$\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \cdots + \sqrt{a_n^2 + b_n^2}$$

represents the total length of the jagged path from A to B in the diagram below.



Since the shortest distance between two points is a straight line, the minimal value $S(X)$ is just the distance from A to B, which is $\sqrt{13^2+x^2}$. Thus $13^2+x^2=y^2$, where both x and y must be integers. This implies that $y^2-x^2=(y-x)(y+x)=13^2$. Factoring, we have either $y-x=1$ and $y+x=169$ or $y-x=y+x=13$. The latter case makes $x=0$ so we reject it. The first case yields $y=85$, $x=84$.

18 The answer is 533. The temperature function is given by

$$T(x,y,z)=(x+y)^2+(y-z)^2.$$

Now consider two similar functions: $U(x,y,z)=T(y,z,x)$ and $V(x,y,z)=T(z,x,y)$. In other words, U and V are obtained from T by cyclically permuting the variables. The average values of T, U and V taken over the surface of the sphere will be the same because of symmetry. Let us call this average value A. Then the average value of $T+U+V$ will equal $3A$. But

$$\begin{aligned}T(x,y,z)+U(x,y,z)+V(x,y,z)&=\\ &(x+y)^2+(y-z)^2+(y+z)^2+(z-x)^2+(z+x)^2+(x-y)^2\\ &=\quad 4x^2+4y^2+4z^2.\end{aligned}$$

However, on the surface of the planet, $4x^2+4y^2+4z^2=4\cdot 20^2=1600$, a constant! Hence the average value of $T+U+V$ will be $1600=3A$.

19 The answer is 350. This problem was deliberately similar to problem 20 of the 1995 Test of Ingenuity. Let us consider the more general problem, where we look at a regular n-gon. The vertices of the n-gon are nth roots of unity $1,\zeta,\zeta^2,\zeta^3,\ldots,\zeta^{n-1}$, which are the n zeros of the polynomial x^n-1. Let us only consider the line segments emanating from the root 1. The product of the lengths of these line segments is

$$|1-\zeta||1-\zeta^2|\cdots|1-\zeta^{n-1}|=|(1-\zeta)(1-\zeta^2)\cdots(1-\zeta^{n-1})|.$$

But recall that

$$x^n-1=(x-1)(x-\zeta)(x-\zeta^2)\cdots(x-\zeta^{n-1}).$$

Divide this by $x-1$,and we get

$$x^{n-1}+x^{n-2}+\cdots+x^2+x+1=(x-\zeta)(x-\zeta^2)\cdots(x-\zeta^{n-1}).$$

Now substitute $x=1$ and we have

$$n=(1-\zeta)(1-\zeta^2)\cdots(1-\zeta^{n-1}),$$

in other words, the product of the lengths emanating from 1 equals n. By symmetryhe product of the lengths emanating from any vertex will equal n. Since there are n vertices, the product will be n^n. However, this includes overcounting by a factor of 2, i.e. we have counted the length from vertex a to vertex b twice in our prouduct. To correct for this, we just take the square root. In other words, the correct answer is $\sqrt{n^n}$. Plugging in $n = 7$, we have $\sqrt{7^7} = 7^3\sqrt{7} = 343\sqrt{7}$.

20 The answer is 159. Consider a sequence with n distinct integers. The ith number ($i = 2, 3, \ldots, n$) will get swapped if and only if it it was the smallest of the first i elements. The probability of this happening is $1/i$. Hence the average number of swaps is just

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$