Test of Ingenuity

Bay Area Math Meet
University of San Francisco
April 26, 1997

Directions: Please do not open the test booklet until the proctor begins the examination. This is a multiple-choice, 20-question exam. You will have one hour to work on the problems. You get one point for a correct answer, zero points for no answer, and $-1/4$ points for an incorrect answer. Because of this penalty for guessing, if you are not sure of the correct answer to a question, it is best not to answer the question.

The questions are arranged in roughly increasing order of difficulty. The last 6 questions are particularly hard and in the event of a tie score, the first prize will go to the student with the highest score on these six problems. Unless you are extremely ambitious, do not attempt all of the problems! Answering just half of them correctly is a very fine achievement! Read the questions first to see which problems are best for you.

Good luck, and have fun!

1 In this problem, all numbers are written in base two. What is 101×101?

A 1010 B 10101 C 11001

D 11010 E 11101

2 Find the area of a triangle with vertices at the points $(4, 2)$, $(3, 0)$ and $(0, 8)$.

A 7 B 8 C $\sqrt{65}$ D 14 E 35

3 Suppose that the graph of $y = f(x)$ is a line with slope 3 while the graph of $y = g(x)$ is a line with slope 1/2, and the graph of $y = h(x)$ is a line with slope 2. Then the graph of $y = g(h(f(h(g(g(x))))))$ is

A a line with slope 6. B a line with slope 3/2.

C a line with slope 24. D a line with slope 3/8.

E none of the above.

4 If Pat is twice as old as Chris was 15 years ago, and Chris is twice as old as Pat was when Chris was born, then how many years ago was Pat twice as old as Chris?

[A] 10 [B] 15 [C] 20 [D] 25 [E] 30

5 Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Find

$$\lfloor 1/10 \rfloor + \lfloor 2/10 \rfloor + \lfloor 3/10 \rfloor + ... + \lfloor 99/10 \rfloor.$$

[A] 395 [B] 420 [C] 450 [D] 485 [E] 495

6 What is the probability of rolling six standard dice and getting six different numbers?

[A] $\frac{1}{6^6}$ [B] $\frac{1}{6^5}$ [C] $\frac{1}{6!}$ [D] $\frac{6!}{6^6}$ [E] $\frac{6}{6!}$

7 What is the remainder when $1 + 19x^{19} + 97x^{97}$ is divided by $x^2 - 1$?

A $116x - 1$ B $116x + 1$ C $2x + 1$ D 1 E 117

8 The plane is divided into squares numbered as shown below. You are travelling in a straight line at constant velocity. At noon, you are at some point in square 15; at 1 P.M., square 16; and at 2 P.M., square 17. What are all the squares in which you might be at 3 P.M.?

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28

A 11, 18, 25 B 11, 12, 18, 19, 24, 25

C 10, 11, 18, 19, 24, 25 D 10, 11, 12, 18, 19, 24, 25, 26

E 10, 11, 12, 17, 18, 19, 24, 25, 26

9 Let C be the circle $x^2 + y^2 = 1$ and let S be the set of the midpoints of all the line segments which have one endpoint on C and another endpoint at $(1997, 1997)$. Then S is

- [A] a circle with radius $\frac{1996}{2}$.
- [B] a semicircle with radius 1.
- [C] a circle with radius 1.
- [D] a circle with radius $\frac{1}{2}$.
- [E] a circle with radius $\frac{\sqrt{2}}{2}$.

10 Put the following three quantities in order from largest to smallest.

$$A = \log_{(\log_2 5)} 7, \quad B = \log_{(\log_3 5)} 7, \quad C = \log_{(\log_3 4)} 7.$$

[A] A, B, C [B] A, C, B [C] B, A, C

[D] C, A, B [E] C, B, A

11 Two balls of radius 1 lie on a flat surface. Their nearest points are at a distance of 87π. Each ball is colored white with a single purple point. At noon, they begin rolling directly towards each other at unknown speeds (which may not be constant). Eventually they meet, and it turns out that the two purple points are touching. If one of the purple points was located at the top of its ball at noon, then at noon, the other purple point was located

A at the top of its ball.

B at the bottom of its ball.

C equidistant from the top and bottom of its ball.

D neither at the top nor at the bottom nor equidistant from the two.

E at any location, depending on how the speeds changed.

12 Evaluate $\dfrac{\sin 50 + \sin 70}{\sin 35 + \sin 55}$, where all angles are in degrees.

A $\sqrt{2}$ B $\sqrt{3}$ C 2 D $\sqrt{6}/2$ E $(\sqrt{6}+\sqrt{2})/2$

13 Compute the infinite series

$$\frac{1}{7} + \frac{3}{7^2} + \frac{5}{7^3} + \frac{7}{7^4} + \frac{9}{7^5} + \cdots.$$

[A] $\frac{11}{49}$ [B] $\frac{1}{4}$ [C] $\frac{2}{7}$ [D] $\frac{2}{9}$ [E] $\frac{6}{25}$

14 Let ABC be an equilateral triangle with side length of 1. Create an infinite set of circles in the following way: First draw the inscribed circle I of this triangle. Then, draw a circle which lies inside triagnle ABC and is tangent both to I and also to to angle A (in other words, to lines AB and AC). Then, draw a new circle which is tangent to the second circle and also to angle A. Continue this process indefinitely, where each new circle is tangent to the previous one and also to angle A. What is the sum of the circumferences of the infinite set of circles?

[A] $\sqrt{6}\pi/4$ [B] $\sqrt{3}\pi/2$ [C] $\sqrt{3}\pi$

[D] 3π [E] Infinite.

15 A random number generator outputs integers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, with each of the eight choices equally likely. If ten such random integers are created, what is the probability that their product is one more than a multiple of 8?

A $\frac{4!}{8^{10}}$ B $\frac{8!}{4^{10}}$ C $\frac{1}{2^{12}}$ D $\frac{1}{2^{10}}$ E $\frac{1}{8}$

16 Mary walks up an escalator which is going up. When she walks at one step per second, it takes her 20 steps to get to the top. If she walks at two steps per second, it takes her 32 steps to get to the top. She never skips over any steps. How many steps does the escalator have?

A 24 B 32 C 64 D 72 E 80

17 How many subsets of the set $\{1, 2, 3, 4, \ldots, 30\}$ have the property that the sum of the elements of the subset is greater than 232?

[A] $\dfrac{2^{30}}{232}$ [B] $30 \cdot 232^2$ [C] 2^{29} [D] $\dfrac{232!}{202! \cdot 30!}$

[E] None of these.

18 Starting with the single point $(3, 17)$, we add a new point each second, by applying any one of the following rules.

1. The point (x, y) "gives birth" to the point $(x + 1, y + 1)$.
2. If x and y are both even, the point (x, y) "gives birth" to the point $(x/2, y/2)$.
3. The pair of points (x, y) and (y, z) "gives birth" to (x, z).

For example, starting with $(3, 17)$, rule #1 yields the new point $(4, 18)$, and then rule #2 yields $(2, 9)$, etc.

Which of the following points will *never* be in our set?

[A] $(1984, 1997)$ [B] $(3, 1998)$ [C] $(1776, 1993)$

[D] $(1969, 1997)$ [E] $(22, 1492)$

19 Let $ABCD$ be a rectangular sheet of paper with $AB = 5$ and $BC = 6$. Corner D is folded up to meet side BC so that the crease of the fold goes through corner A. If the crease meets CD at E, then the measure of $\angle AED$ is

A $\arctan\left(\frac{6}{5}\right)$ B $\frac{\pi}{2} - \frac{1}{2}\arcsin\left(\frac{5}{6}\right)$ C $\frac{1}{2}\arcsin\left(\frac{5}{6}\right)$

D $\arcsin\left(\frac{3}{5}\right)$ E $\frac{\pi}{2} - \frac{1}{2}\arcsin\left(\frac{6}{\sqrt{61}}\right)$

20 A *great circle* is a circle drawn on a sphere whose center is also the center of the sphere. There are 8 great circles on a sphere, no three of which meet at any point. They divide the sphere into how many regions?

A 58 B 62 C 64 D 128 E 256

Answers

1. C
2. A
3. B
4. E
5. C
6. D
7. B
8. E
9. D
10. E
11. A
12. D
13. D
14. B
15. C
16. E
17. C
18. A
19. B
20. A