Key to Test of Ingenuity **Bay Area Math Meet** **April 20, 1997**

The median score was approximately 5 points out of a maximum of 20. Many students lost points by guessing incorrectly on the harder problems. As you can see from the table below, almost half of the students guessed incorrectly at problem #20. Next year, beware: problem #20 is *supposed* to be hard! The "obvious" guess is almost surely not the correct answer!

question #	1	2	3	4	5	6	7	8	9	10
% correct	54.2	47.9	75.0	48.6	56.3	61.1	30.6	28.5	41.7	52.8
% blank	19.4	19.4	9.7	18.8	15.3	8.3	43.1	15.3	19.4	11.8
% wrong	26.4	32.6	15.3	32.6	28.5	30.6	26.4	56.3	38.9	35.4

question #	11	12	13	14	15	16	17	18	19	20
% correct	37.5	25.0	16.0	18.8	7.6	27.8	5.6	7.6	9.0	5.6
% blank	21.5	56.9	42.4	49.3	63.2	42.4	69.4	59.7	75.0	45.1
% wrong	41.0	18.1	41.7	31.9	29.2	29.9	25.0	32.6	16.0	49.3

1 The answer is 11001. 101 in base 2 is equal to $1 \cdot 1 + 0 \cdot 2 + 1 \cdot 2^2 = 5$ in base ten. Hence the product is $5 \times 5 = 25 = 16 + 8 + 1$.

2 The answer is 7. Just draw a rectangle with sides parallel to the axes which encloses the triangle. The area of the triangle is just the area of the rectangle minus the areas of the shaded triangles, or $4 \cdot 8 - \frac{1}{2} 4 \cdot 6 - \frac{1}{2} 3 \cdot 8 - \frac{1}{2} 1 \cdot 2$.

3 The answer is 3/2. It is easy to check that if $f(x) = ax + b$ and $g(x) = cx + d$, then $f(g(x)) = a(cx + d) + b = acx + ad + b$, in other words, when you compose the functions, the new slope is just the product of the two slopes. So the answer is just the product of all the slopes.

4 The answer is 30. Let p and c, respectively, denote the ages of Pat and Chris today. Then we have $p = 2(c - 15)$ and $c = 2(p - c)$. Solving this system yields $p = 90$, $c = 60$. If we let x denote the number of years ago when Pat was twice as old as Chris, we have $90 - x = 2(60 - x)$, and $x = 30$.

5 The answer is 450. We have

$$\left\lfloor \frac{k}{10} \right\rfloor = \begin{cases} 0 & \text{for } k = 1, 2, \ldots, 9; \\ 1 & \text{for } k = 10, 11, \ldots, 19; \\ 2 & \text{for } k = 20, 21, \ldots, 29; \\ \vdots & \qquad \vdots \\ 9 & \text{for } k = 90, 91, \ldots, 99. \end{cases}$$

Therefore the sum in question is equal to

$$0 \cdot 9 + 1 \cdot 10 + 2 \cdot 10 + \cdots 9 \cdot 10 = 10(1 + 2 + 3 + \cdots + 9) = 10(9)(10)/2.$$

6 The answer is $6!/6^6$. There are $6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 6^6$ possible ways of rolling six dice. If the six numbers are all different, then they are a permutation of the numbers 1, 2, 3, 4, 5, 6. There are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$ such permutations.

7 The answer is $116x + 1$. We have

$$\frac{1 + 19x^{19} + 97x^{97}}{x^2 - 1} = q(x) + \frac{ax + b}{x^2 - 1},$$

where $q(x)$ is the quotient polynomial and $ax + b$ the remainder. Multiplying by $x^2 - 1$, we have

$$1 + 19x^{19} + 97x^{97} = q(x)(x^2 - 1) + ax + b.$$

Now we can substitute $x = 1$ and $x = -1$ which yields $117 = a + b$, $-115 = -a + b$ and we can easily solve for a and b. However, there is an easier way to do this problem, which will make sense if you know a little number theory. We are trying to find the remainder modulo $x^2 - 1$. In the modulo $x^2 - 1$ universe, we can reduce x^2 to 1! Hence $1 + 19x^{19} + 97x^{97}$ reduces (modulo $x^2 - 1$) to $1 + 19x + 97x = 1 + 116x$.

8 The answer is 10, 11, 12, 17, 18, 19, 24, 25, 26. The problem is pretty complicated unless one separates the motion into its horizontal and vertical components. Then it is easy to see that vertical component of the position at 3 P.M. will be at most 1 square away from home, while the horizontal component could be zero, one or two squares away.

9 The answer is a circle with radius $\frac{1}{2}$. If you are familiar with geometric tranformations, it is immediately obvious that S is just the image of the unit circle by a **homothety** with scale factor 1/2. However, most students don't know about homotheties, so here is a simple analytic geometry solution. The parametric form of the unit circle is $(\cos t, \sin t)$ as t ranges from 0 to 2π. Thus the set S is given parametrically by

$$\left(\frac{\cos t + 1997}{2}, \frac{\sin t + 1997}{2} \right) = \left(\frac{\cos t}{2}, \frac{\sin t}{2} \right) + \left(\frac{1997}{2}, \frac{1997}{2} \right),$$

which is a circle with radius $\frac{1}{2}$ and center $(\frac{1997}{2}, \frac{1997}{2})$. We urge you to learn about geometric transformations such as homotheties. Some great books to read are *Geometry Revisited* by Coxeter and Greitzer, *Geometric Transformations* by Yaglom, and *Geometry* by Coxeter. The first two are inexpensive paperbacks published by the Mathematical Association of America under the New Mathematical Library series. The third book is a longer, harder read, but is a classic, published by Wiley.

10 The answer is C, B, A. Note that if $x, y, z > 0$ and $x > y$ then $\log_x z < \log_y z$. Thus $\log_2 5 > (\log_3 5)$, so $A < B$. But also, $\log_3 4 < \log_3 5$ which implies that $B < C$.

11 The answer is that the other purple point was located at the top of its ball. The total rotation of the balls is $43\frac{1}{2}$ complete turns. The ball that we know about turned an integral number of turns, plus one quarter turn. The other ball is turning in the opposite direction. It has to turn an integral number of full turns, plus a quarter turn (in this opposite direction). So its purple point had to be located on top as well.

12 The answer is $\sqrt{6}/2$. We have

$$\begin{aligned}\frac{\sin 50 + \sin 70}{\sin 35 + \sin 55} &= \frac{\sin(60-10) + \sin(60+10)}{\sin(45-10) + \sin(45+10}\\ &= \frac{\sin 60 \cos 10 - \sin 10 \cos 60 + \sin 60 \cos 10 + \sin 10 \cos 60}{\sin 45 \cos 10 - \sin 10 \cos 45 + \sin 45 \cos 10 + \sin 10 \cos 45}\\ &= \frac{2 \sin 60 \cos 10}{2 \sin 45 \cos 10}\\ &= \frac{\sin 60}{\sin 45}.\end{aligned}$$

13 The answer is 2/9. Compute the infinite series

$$\frac{1}{7} + \frac{3}{7^2} + \frac{5}{7^3} + \frac{7}{7^4} + \frac{9}{7^5} + \cdots.$$

Just employ the standard trick for summing an ordinary geometric series. Let the unknown sum be x. Then we have

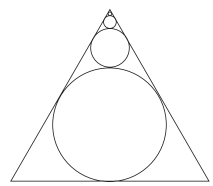
$$7x = 1 + \frac{3}{7} + \frac{5}{7^2} + \frac{7}{7^3} + \frac{9}{7^4} + \cdots.$$

Subtracting the original sum from this yields

$$6x = 1 + \frac{2}{7} + \frac{2}{7^2} + \frac{2}{7^3} + \frac{2}{7^4} + \cdots = 1 + \frac{\frac{2}{7}}{1 - \frac{1}{7}} = 1 + \frac{1}{3}.$$

Now solve for x.

14 The answer is $\sqrt{3}\pi/2$. Let ABC be an equilateral triangle with side length of 1. Let the diameters of the circles be $d_1, d_2, \ldots$. The circles form a tower which "fills up" the triangle, hence $d_1+d_2+\cdots$ is equal to the altitude of the triangle, which is $\sqrt{3}/2$. The sum of the circumferences is just π times this.



15 The answer is $1/2^{12}$. Clearly, all ten numbers have to be odd. The probability of this is just $1/2^{10}$. But if all ten numbers are odd, the product could have remainders of 1, 3, 5 or 7 upon division by 8. Each of these 4 possibilites are equally likely, so we must divide $1/2^{10}$ by 4 to get the correct answer. Here is a more rigorous argument that all 4 possibilities are equally likely: Let a be odd, and consider the numbers $a, 3a, 5a, 7a$ modulo 8. They are distinct, for if, say $3a \equiv 7a \pmod{8}$, then we would have $4a \equiv 0 \pmod{8}$, which is impossible, since a is odd. Now let $a_1, a_2, \ldots, a_{10}$ be a sequence of odd random numbers with $a_1 a_2 \cdots a_{10} \equiv 1 \pmod{8}$. Then

$$(3a_1), a_2, a_3, \ldots, a_{10}; \quad (5a_1), a_2, a_3, \ldots, a_{10}; \quad (7a_1), a_2, a_3, \ldots, a_{10}$$

are three different sequences with products congruent to 3, 5 and 7 modulo 8, respectively. In other words, we have shown that for each sequence with product congruent to 1, there are 3 other sequences with the three other possible values for the product. Hence exactly 1/4 of the sequences have a product congruent to 1.

16 The answer is 80. Let n be the number of steps. In 20 seconds, Mary walks up 20 steps and gets to the top. Consequently the escalator moves at a speed of $n-20$ steps in 20 seconds. Likewise, it moves $n-32$ steps in 16 seconds. Equating these two, we have

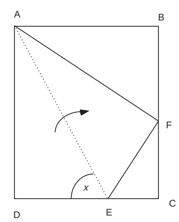
$$\frac{n-20}{20} = \frac{n-32}{16},$$

which is easy to solve.

17 The answer is 2^{29}. Notice that the sum of all the elements of the set $S = \{1, 2, 3, \ldots, 30\}$ is $1+2+3+\cdots+30 = 30 \cdot 31/2 = 465$. Let A be a subset of S and let A^c denote the *complement* of A (the elements of S which are not elements of A. The sum of the elements of A plus the sum of the elements of A^c must equal 465. Because $465 = 232 + 233$, if the sum of the elements of a A is greater than 232, the sum of the elements of A^c must be less than 232. In other words, there is a one-to-one correspondence between subsets whose element sum is greater than 232 and subsets whose element sum is not (namely, $A \leftrightarrow A^c$). Hence the number of subsets whose element sum is greater than 232 is exactly half of the total number of subsets of S, and the number of subsets of S is 2^{30}. Note that this problem is really very similar to problem #15.

18 The answer is $(1984, 1997)$. Notice that $17 - 3 = 14$ is a multiple of 7, and that all new points (x, y) that are "born" will also have the property that $y - x$ is a multiple of 7. For all of the other choices the difference of the coordinates was a multiple of 7.

19 The answer is $\dfrac{\pi}{2} - \dfrac{1}{2}\arcsin\left(\dfrac{5}{6}\right)$. In the diagram below, the crease is line segment AE and the point F is where D touches side BC when the paper is folded. Let $x = \angle AED$.



Triangle AFE is the "folded" position of triangle ADE, so the two triangles are congruent. In particular, $AF = 6$ and $\angle AEF = x$. Notice that triangles ABF and FCE are similar. Thus $\angle FEC = \angle BFA$. But $\angle FEC = \pi - 2x$, so

$$5/6 = AB/AF = \sin \angle BFA = \sin(\pi - 2x).$$

Consequently, $\pi - 2x = \arcsin(5/6)$, so $x = (\pi - \arcsin(5/6))/2$.

20 The answer is 58. Let $f(n)$ denote the number of regions that n great circles divide the sphere into. It is easy to check that $f(1) = 2$, $f(2) = 4$, $f(3) = 8$, which led many students to erroneous guesses such as $f(n) = 2^n$, because it is extremely hard to draw a picture to compute even $f(4)$. Let us attempt to compute $f(4)$ without a picture. Start with 3 great circles and 8 regions. The 4th great circle must intersect each of the three previous great circles (since great circles always meet), and it must intersect each of these in 2 points. Imagine drawing this 4th great circle, starting at one of these points of intersection. Keep a running count of the new regions added as you continue drawing the great circle. Each time you intersect one of the old great circles, you can add a new region. You intersect 5 times, and then finally join up with your starting point, adding a total of 6 new regions. Thus $f(4) = 8 + 6 = 14$, and in general, $f(n+1) = f(n) + 2n$. Continuing recursively, we easily compute

$$f(8) = 14 + f(7) = 14 + 12 + f(6) = \cdots = 14 + 12 + \cdots + 2 + f(1) = 58.$$