**Key to Test of Ingenuity** **Bay Area Math Meet** **April 25, 1998**

The median score was approximately 6 points out of a maximum of 20. Many students lost points by guessing incorrectly on the harder problems. Beware: for the later problems, the “obvious” guess is almost surely *not* the correct answer!

| question # | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| % correct | 96.4 | 69.8 | 83.4 | 56.8 | 58.6 | 55.6 | 45.0 | 56.8 | 49.7 | 62.1 |
| % blank | 1.2 | 10.7 | 6.5 | 14.8 | 8.3 | 25.4 | 20.7 | 2.4 | 30.8 | 16.0 |
| % wrong | 2.4 | 19.5 | 10.1 | 28.4 | 33.1 | 18.9 | 34.3 | 40.8 | 19.5 | 21.9 |

| question # | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| % correct | 11.8 | 19.5 | 25.4 | 20.1 | 10.7 | 9.5 | 14.8 | 8.3 | 3.6 | 5.9 |
| % blank | 45.6 | 43.2 | 22.5 | 40.8 | 41.4 | 46.7 | 65.7 | 67.5 | 78.7 | 71.6 |
| % wrong | 42.6 | 37.3 | 52.1 | 39.1 | 47.9 | 43.8 | 19.5 | 24.3 | 17.8 | 22.5 |

**1** The answer is $16\pi$. Let $r, C$ respectively denote the radius and circumference. $C = 2\pi r = 8\pi$ implies that $r = 4$. The area is $\pi r^2 = 16\pi$.

**2** The answer is 90. There are 9 choices for the first digit (1–9) and there are ten choices for the second digit (0–9).

**3** The answer is $4xy$.

$$f(x+y, x-y) = (x+y)^2 - (x-y)^2 = (x^2 + 2xy + y^2) - (x^2 - 2xy + y^2).$$

**4** The answer is 60/13. Since $5^2 + 12^2 = 13^2$, it is a right triangle, so the area is $5 \cdot 12/2$. If $h$ is the length of the desired altitude, we have $13h/2 = 5 \cdot 12/2$.

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**5** The answer is 2009. A normal year has 365 days, and 365 has a remainder of 1 when divided by 7. That means that if April 25 is day $x$ this year, then next year it will be day $x+1$. The only exception to this is if the next year is a leap year, in which case the new day will be $x+2$. Now it is just a matter of careful counting.

**6** The answer is 26. Let $e, s$ be the current ages of Eric and Samuel, respectively. The first sentence yields $e-8=4(s-8)$, while the second yields $s+2=2(e-s)$. Add the two equations, and solve for $e+s$.

**7** The answer is 27. Every door connects exactly two rooms, with the exception of the single entrance door. Hence, if we add up the number of doors in each room, the sum will be odd (since we are double-counting every door except the entrance door). This sum will also be $3n$. Hence $n$ must be odd.

**8** The answer is "It is impossible." To have an average speed of 60 mph for a 2-mile trip, the time required is 2 minutes. But two minutes have already elapsed!

**9** The answer is 1. Use the fact that for each $k$,

$$\frac{1}{k^2+k}=\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}.$$

Now we employ the "telescope" trick: For each $n$,

$$\frac{1}{1^2+1}+\frac{1}{2^2+2}+\cdots+\frac{1}{n^2+n}=\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n}-\frac{1}{n+1}.$$

All the terms cancel except for the first and the last, yielding a sum of $1=\frac{1}{n+1}$. Certainly, as $n$ tends to infinity, this sum approaches 1.

**10** The answer is 43. Since 43 is not a multiple of 3, if it were a Nuggitz number, then it would need at least one 20-nugget bucket. But neither $43-20$, nor $43-40$ will work. So 43 is not a Nuggitz number. On the other hand, it is easy to check that 44–49 are Nuggitz numbers (for example, $47=20+3\cdot 9$), and thereafter, we can just add a 6-nugget bucket to yield 50–55, etc. So 43 is the largest non-Nuggitz number.

**11** The answer is 18. What the question is asking, using more sophisticated language, is "What is the order of a permutation of 10 objects?" The order of a permutation is the least number of times it must be repeated to yield the identity permutation. For example, consider the permutation

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3 | 7 | 4 | 5 | 1 | 6 | 2 | 9 | 8 | 10 |

which takes 1 to 3, 2 to 7, etc. How do we figure out its order? Start with 1; it goes to 3. Now, where does 3 get sent? To 4. Then 4 goes to 5, and 5 goes to 1. In other words, we have the *cycle* (1345) of length 4. If we repeated the permutation 4 times, we would be assured that 1, 3, 4, and 5 would be back in their original places. To figure out what happens to the rest of the numbers, just keep extracting cycles. Verify that this permutation consists of the *disjoint cycles* (1345), (6), (72), (89), and (10). The orders of these cycles are respectively 4, 1, 2, 2, 1, so the order of the entire permutation must be 4, the least common multiple of these numbers.

For any permutation, we can decompose it into disjoint cycles, and the order of the permutation will be the LCM of the cycle lengths. In our problem, the lengths have to add up to 10. We can get order 7 with cycles of length 7, 1, 1, 1; likewise we can get order 14 with cycles of length 7, 2, 1. Cycles of length 7 and 3 yield a permutation with order 21, while cycles of length 2, 3, 5 yield a permutation with order 30. But to get a permutation of order 18, we need a cycle of length 9 as well as a cycle of length 2, which is impossible ($9 + 2 > 10$).

**12** The answer is 8. The points in the plane satisfying $|x| + |y| \leq 1$ form a diamond centered at the origin with vertices at $(0, \pm 1)$, $(\pm 1, 0)$. The area of this diamond is 2. Now, if we replace $x$ and $y$ with $x - 1$ and $y - 1$ respectively, the region will be translated to have a center of $(1, 1)$, but will still be a diamond with area 2 (but with vertices at $(0, 0)$, $(1, 0)$, $(2, 1)$, $(1, 2)$, $(0, 1)$). To get the region $R$ described in the problem, we now need to replace $x$ and $y$ respectively with $|x|$ and $|y|$. This new region contains all the points of the old region, except the sign of $x$ or $y$ can be arbitrary. Since the diamond was only in quandrant I, the region $R$ consists of all 4 reflections of this diamond into the other three quandrants, a total of 4 diamonds with area $4 \cdot 2$.

**13** The answer is 200. The base of each triangular roof panel is of course 10. But observer sees an equilateral triangle, so the slant height of each panel is also 10!.

**14** The answer is 1,998. Let $b_t$, $v_t$ denote the bacterial and virus populalations, respectively, at time $t$. We have $b_0/v_0 = 1998$. Observe that

$$\frac{b_{t+1}}{v_{t+1}} = \frac{2(b_t - v_t)}{2v_t} = \frac{b_t}{v_t} - 1.$$

Clearly the quotient $b_t/v_t$ will equal 0 when $t = 1998$.

**15** The answer is $42/125$. We shall first consider the probability that the product is *not* a multiple of 6. Let $E$, $T$ denote, respectively, the events that the product of the three numbers is not even and is a not multiple of three. Then the event $(E \cup T)$ is equivalent to the event that the product is not a multiple of 6. The number of elements in this event (let the cardinality of a set $S$ be denoted by $|S|$) is

$$|E| + |T| - |E \cap T| = 3^3 + 4^3 - 2^3 = 27 + 64 - 8 = 83.$$

Hence the probability that the product is a multiple of 6 will be $1 - 83/125 = 42/125$.

**16** The answer is 51. We break it down into three cases:

1. The selection has 5 colors. In this case, there is just one (distinguishable) way to pick the balls.
2. The selection has 4 colors. In this case, there are two ball of one color and 3 balls each of one of the other colors. There are 5 ways to choose the "double-up" color, and $\binom{4}{3} = 4$ ways to pick the other 3 colors, a total of $5 \cdot 4 = 20$ ways.
3. The selection has 3 colors. Now we double up two colors (in $\binom{5}{2} = 10$ ways) and pick one of the remaining 3 colors for a single ball (3 ways). So the total is $10 \cdot 3 = 30$.

There are no other cases, since there cannot be less than 3 colors chosen (otherwise, we would have to have more than 2 balls of one color).

**17** The answer is 25,366,109. Consider the equation $x^3 - y^3 = p$, where $x$ and $y$ are positive integers and $p$ is a prime. Since the left-hand side factors into $(x - y)(x^2 + xy + y^2)$, but the right-hand side is a prime, the only possibility is that $x - y = 1$. Substituting $x = y + 1$, we get $p = (y+1)^3 - y^3 = 3y^2 + 3y + 1$. Hence $p$ must have a remainder of 1 when divided by 3. It is easy to compute these remainders; just add the digits.

**18** The answer is 28. Verify the following facts:

- $\sigma(ab) = \sigma(a)\sigma(b)$ if $a$ and $b$ are relatively prime (share no factors other than 1).
- If $n$ is a power of 2, then $\sigma(n)$ is odd.
- If $n$ is an odd square, then $\sigma(n)$ is odd.
- If $n$ is odd, but not a square, then $\sigma(n)$ is even.

These three facts allow us to conclude that $\sigma(n)$ is even *if and only if* $n = 2^a u^2$ where $a \geq 0$ and $u$ is odd. Now it is a matter of careful counting ($u$ is at most 17, etc.)

**19** The answer is $\sqrt{6}/3$. The inequality rearranges into $x^3 + y^3 \geq 3x^2y + 3xy^2$. Factoring yields $(x+y)(x^2 - xy + y^2) \geq 3xy(x+y)$. Since $x + y = -1$, we have $-(x^2 - xy + y^2) \geq -3xy$, or $x^2 + 2xy + y^2 \leq 6xy$. Once again, substituting $x + y = -1$ yields $(-1)^2 \leq 6xy$, so $xy \geq 1/6$. The points in the plane which satisfy this inequality are two disjoint regions, one in quadrant I and one in quadrandt III, namely the "interiors" of the hyperbola $xy = 1/6$. Thus the line segment that we seek is that subset of the line $x + y = -1$ which satisfies $xy = 1/6$. By symmetry, the coordinates of the endpoints will be $(a, b)$ and $(b, a)$, where $a$ and $b$ are the roots of system $x + y = -1$, $xy = 1/6$. The length of the segment is $\sqrt{(a-b)^2 + (b-a)^2} = \sqrt{2}|a - b|$. It is easy to find $|a - b|$, since $a + b = -1$ and $ab = 1/6$. We have $(a-b)^2 = (a+b)^2 - 4ab = 1 - 2/3 = 1/3$. So our answer is $\sqrt{2}/\sqrt{3} = \sqrt{6}/3$.

*Note*: There is a very elegant alternate solution to this problem, using complex numbers. Can you find it?

**20** The answer is $1681/3281$. Let $s_k, r_k$ respectively denote the events sunny on April $k$, rainy on April $k$, and let $\vee$ denote logical "and." We must compute $P(s_9 \vee s_5 \vee s_1)/P(s_9 \vee s_1)$. Let us expand outward from April 1: We are given that $P(s_2 \vee s_1) = 2/3$ and $P(r_2 \vee s_1) = 1/3$. A sunny day on April 3 either follows a sunny April 2 or a rainy April 2; hence

$$P(s_3 \vee s_1) = \frac{2}{3}P(s_2 \vee s_1) + \frac{1}{3}P(r_2 \vee s_1) = \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{5}{9}.$$

This actually tells us more: the probability is $5/9$ that the weather on April $k$ is equal to the weather on April $k+2$. Using this, we now compute the probability that it is sunny on April 5, given that it was sunny on April 1. Either April 3 was sunny or was rainy. The total probability that April 5 is sunny, then, is

$$\frac{5}{9}P(s_3 \vee s_1) + \frac{4}{9}P(r_3 \vee s_1) = \left(\frac{5}{9}\right)^2 + \left(\frac{4}{9}\right)^2 = \frac{41}{81}.$$

Likewise, we see that

$$P(s_9 \vee s_1) = \left(\frac{41}{81}\right)^2 + \left(\frac{40}{81}\right)^2$$

and

$$P(s_9 \vee s_5 \vee s_1) = \left(\frac{41}{81}\right)^2.$$