# Test of Ingenuity

BAY AREA MATH MEET
UNIVERSITY OF SAN FRANCISCO
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**Directions**: Please do not open the test booklet until the proctor begins the examination. This is a multiple-choice, 20-question exam. You will have one hour to work on the problems. You get one point for a correct answer, zero points for no answer, and $-1/4$ points for an incorrect answer. Because of this penalty for guessing, if you are not sure of the correct answer to a question, it is best not to answer the question.

The questions are arranged in roughly increasing order of difficulty. The last 6 questions are particularly hard and in the event of a tie score, the first prize will go to the student with the highest score on these six problems. Unless you are extremely ambitious, do not attempt all of the problems! Answering just half of them correctly is a very fine achievement! Read the questions first to see which problems are best for you.

Good luck, and have fun!

**1** Find the perimeter of a square with area 100.

A 20 B 40 C 50 D 100 E 400

**2** Compute the following sum. The values, including your answer, are written in base-5.

$$121 + 44 =$$

A 210 B 212 C 213 D 220 E 223

**3** Let $f(x, y) = x^2 + y^2$. Then $f(x + y, x - y) =$

A $4xy$ B $x^2 + y^2$ C $2x^2 + 2y^2$ D $0$ E $2xy$

**4** A Klopstockian license plate consists of a number between 1 and 8, inclusive, followed by three letters, with no repeating letters allowed. Thus "7XYZ" is legal, but "7XYX" is not. How many legal Klopstockian license plates are possible?

A $26 \cdot 25 \cdot 24$ B $8 \cdot 26^3$ C $8 \cdot 26!$ D $8 \cdot 26 \cdot 25 \cdot 24$

E $8^3 \cdot 26!$

**5** Last year I took 64 road-trips. On exactly half of the road-trips, I made at least one stop at a restaurant. Exactly half of my stops at restaurants were at diners. On how many trips did I stop at a diner?

A The only possible answer is 8.

B The only possible answer is 16.

C The only possible answer is 32.

D The answer can be any integer from 1 to 64.

E None of the above.

**6** Annie and Betty ran a 100-foot race. Each person ran at a steady rate, and Annie reached the finish line 5 feet ahead of Betty. They decided to race again, and in an attempt to be fair, Annie started 5 feet behind the original starting line. If they again ran at the same steady rates that they ran during the first race, what was be the outcome of the second race?

A They tie. B Annie wins by 3 in. C Betty wins by 3 in.

D Annie wins by 6 in. E None of the above.

**7** Let $ABCD$ be a square with center at $X$ and side length 8. Let $XYZ$ be a right triangle with right angle at $X$ with $XY = 10$, $XZ = 24$. If $XY$ intersects $BC$ at $E$ such that $CE = 2$ and $EB = 6$, find the area of the region which is common to the triangle and the square.

A 13 B 16 C $4\sqrt{2}$ D $3(\sqrt{6}+\sqrt{2})$

E $26-4\sqrt{6}$

**8** Describe the locus of points $(x, y)$ in the plane which satisfy the equation

$$4x^2 - 4xy + y^2 = 1.$$

[A] An ellipse. [B] Two intersecting lines. [C] A hyperbola.

[D] Two intersecting parabolas. [E] Two parallel lines.

**9** A fan with four equally spaced blades is running at 100 revolutions per second clockwise. A movie camera films this fan, shooting 48 frames per second. When the film is played (at the rate of 48 frames per second), what will be the apparent speed of the fan?

[A] 0 revolutions per second.

[B] 4 revolutions per second counterclockwise.

[C] 4 revolutions per second clockwise.

[D] 52 revolutions per second clockwise.

[E] 2.08 revolutions per second clockwise.

**10** Let $a$ and $b$ be distinct positive real numbers. Let

$$X = \frac{a+b}{2}, \quad Y = \sqrt{ab}, \quad Z = \frac{2}{\frac{1}{a} + \frac{1}{b}}.$$

List these three values in increasing order.

A $Z, Y, X$ B $Y, X, Z$ C $X, Z, Y$

D It depends on $a$ and $b$. E None of the above.

**11** During a 3-minute time interval, several people observed a worm crawling from east to west in a straight line. Each observer watched the worm for exactly one minute, and the worm was always watched by at least one person. It turns out that each observer saw the worm move forward exactly one foot to the west. To the nearest inch, what is the greatest distance that the worm could have traveled during the 3 minutes?

A 36 B 48 C 60 D 72 E 84

**12** How many ordered pairs $(x, y)$ of integers are solutions to

$$\frac{xy}{x+y} = 99?$$

$\boxed{\text{A}}$ 12 $\boxed{\text{B}}$ 15 $\boxed{\text{C}}$ 29 $\boxed{\text{D}}$ 30 $\boxed{\text{E}}$ 49

**13** Let $ABC$ be a triangle, with points $P$ and $Q$ respectively lying on line segments $AB$ and $BC$, such that

$$\frac{AP}{PB} = 3 \quad \text{and} \quad \frac{BQ}{QC} = 1.$$

If lines $AQ$ and $CP$ intersect at $R$, compute $AR/RQ$.

$\boxed{\text{A}}$ $9/2$ $\boxed{\text{B}}$ 5 $\boxed{\text{C}}$ $11/2$ $\boxed{\text{D}}$ 6 $\boxed{\text{E}}$ $19/3$

**14** A spherical ball with radius 1 inch is sitting on the floor and touching the wall. Let $F$ and $W$ be the points where the ball touches the floor and wall, respectively. A tiny light is on the floor at the point $L$, which is 4 inches from the wall, and triangle $LFW$ is perpendicular to the floor. What is the highest point (in inches) above the floor that is darkened by the shadow of the ball? Assume, of course, that the wall and floor are perpendicular.

[A] 3 [B] 4 [C] 5 [D] $3\sqrt{3}-\sqrt{5}$ [E] $2\sqrt{3}-\sqrt{5}/2$

**15** The polynomial $(x+2y+3z)^{20}$ is multiplied out, and all the like terms are collected. How many terms will there be?

[A] $3^{20}-2^{20}$ [B] $\frac{20!}{2!3!}$ [C] $21 \cdot 21$ [D] $21 \cdot 20$

[E] $21 \cdot 11$

**16** Find the minimum value of

$$(u-v)^2+\left(\sqrt{2-u^2}-\frac{9}{v}\right)^2$$

for $0 < u < \sqrt{2}$ and $v > 0$.

[A] 1 [B] 2 [C] 8 [D] $4\sqrt{2}$ [E] $9\log_e 3$

**17** If $x^2+y^2+z^2=49$ and $x+y+z=x^3+y^3+z^3=7$, find $xyz$.

[A] $-112$ [B] $-84$ [C] $343/27$ [D] 7 [E] $49/3$

**18** Let $x$ and $y$ be positive numbers such that $x^{\log_y x} = 2$ and $y^{\log_x y} = 16$. Solve for $x$.

A 2 B $\sqrt[3]{4}$ C $2^{\sqrt{2}}$ D $2^{\sqrt[3]{2}}$ E $2^{\sqrt[3]{4}}$

**19** Let $P = A_1A_2 \cdots A_{20}$ be a regular 20-gon which is inscribed in a circle of radius 1. Let $S$ be the set containing the edge $A_1A_2$ and all the diagonals of $P$ which are parallel to this edge, including edge $A_{11}A_{12}$. Find the sum of the lengths of the elements of $S$.

A $2\csc 9^\circ$ B $40\sin 18^\circ$ C $40\tan 18^\circ$ D $4\cot 18^\circ$

E $12\sec 9^\circ$

**20** Eleven points are chosen randomly on the surface of a sphere. What is the probability that all eleven points lie on some hemisphere of this sphere?

A 11/2048

B 7/128

C 1/2

D 63/8192

E 23/256