Solutions to the Test of Ingenuity Bay Area Math Meet April 17, 1999

The median score was approximately 6 points out of a maximum of 20. Many students lost points by guessing incorrectly on the harder problems, especially #20. Beware: for the later problems, the "obvious" guess is almost surely *not* the correct answer!

question #	1	2	3	4	5	6	7	8	9	10
% correct	99	60	86	86	68	57	56	36	31	36
% blank	0	33	7	4	2	6	32	26	21	5
% wrong	1	7	7	10	30	36	11	39	48	59

question #	11	12	13	14	15	16	17	18	19	20
% correct	11	6	13	16	24	6	12	6	4	2
% blank	17	76	70	67	57	84	70	81	90	74
% wrong	71	18	17	17	19	10	18	12	6	24

1 The answer is $\boxed{40}$ If x is the side length of the square, we have $x^2 = 100$, so $x = 10$, and the perimeter is $4x$.

2 The answer is $\boxed{220}$ In base 5, the value 5 is written "10", so when we do the addition, the units digits sum to $1 + 4 = 10$, which we write as "0", carrying the 1. The 5's digits (not 10's digits!) sum to $2 + 4$ plus the carry of 1 to equal 7 in base 10 which is 12 in base 5, so we write a "2" and carry 1 to the 25's digits. These sum to 1 plus the carry of 1, or 2.

3 The answer is $\boxed{2x^2 + 2y^2}$ We have

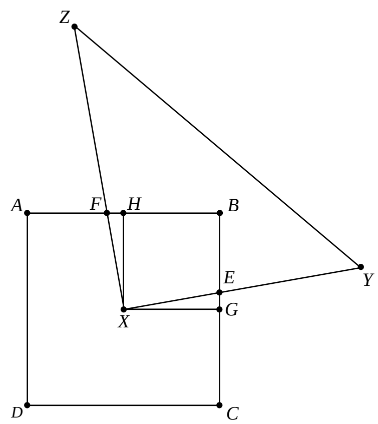
$$f(x + y, x - y) = (x + y)^2 + (x - y)^2 = x^2 + 2xy + y^2 + x^2 - 2xy + y^2.$$

4 The answer is $\boxed{8 \cdot 26 \cdot 25 \cdot 24}$ There are 8 choices for the number, and 26 choices for the first letter, but only 25 for the second letter (to eliminate repeats) and likewise 24 choices for the third letter.

5 The answer is $\boxed{\text{“None of the above.”}}$ Certainly, on 32 road trips, I visited at least one restaurant, but it is possible that I visited many restaurants. For example, perhaps I visited 10 restaurants on one trip. Then I would have visited 5 diners on that trip alone. Thus the number of trips which featured stops at diners could range from a minimum of 16 to a maximum of 32 (if on each of the 32 trips I visited at least 2 restaurants).

6 The answer is $\boxed{\text{Annie wins by 3 in}}$ Betty can run 95 feet during the time it takes Annie to run 100 feet. So in the second race, the two runners will both reach the point that is 5 feet from from the finish line at the same time. Then, the remainder of the race will just like the first race (i.e., Annie will win), only by $5/100 = 1/20$th as much distance.

7 The answer is $\boxed{16}$ No matter how the triangle XYZ rotates about the point X, the area common to the triangle XYZ and the square $ABCD$ will not change. This follows from the fact that triangles XHF and XGE are congruent (because angles ZXY, HXG, XGE, XHF are all right angles, and $XG = XH$). Hence the area is equal to the area of the square $HXGB$.



8 The answer is $\boxed{\text{two parallel lines}}$ Notice that $4x^2 - 4xy + y^2 = (2x - y)^2$. Hence we have $(2x - y)^2 = 1$, so the locus consists of the lines $2x - y = 1$ and $2x - y = -1$.

9 The answer is $\boxed{\text{4 rev per sec clockwise}}$ Imagine that one blade is at 12 o'clock when the frame is shot. The next frame is shot in 1/48 of a second. This blade will have made $100/48 = 2\frac{1}{12}$ rotations, but all that is perceived is the fractional part of this number, i.e., the shot will show the fan blade at 1 o'clock. Thus the perceived speed is 1/12 of a revolution clockwise per 1/48 of a second, or 4 revolutions clockwise per second.

10 The answer is $\boxed{Z, Y, X}$ The important **arithmetic-geometric mean inequality**, also known as AM-GM, states that if a and b are positive real numbers, then

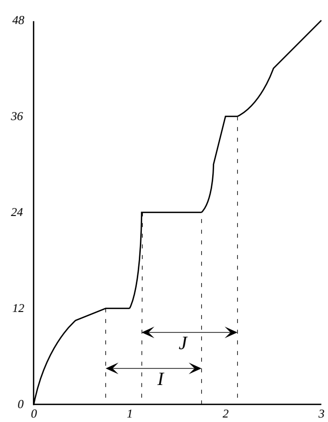
$$\frac{a+b}{2} \geq \sqrt{ab},$$

with equality holding if and only if $a = b$. There are many ways to prove this; one algebraic method starts with the fact that $(\sqrt{a} - \sqrt{b})^2 \geq 0$. For a fuller discussion (including generalizations and geometric methods) consult *Analytic Inequalities* by Nicholas Kazarinoff or *The Art and Craft of Problem Solving* by Paul Zeitz. Anyway, AM-GM immediately implies that $X \geq Y$. And since Z is the reciprocal of the average of $1/a$ and a/b, AM-GM implies (remember, if $u \geq v$ and $u, v > 0$, then $1/u \leq 1/v$) that

$$Z \leq \frac{1}{\sqrt{\frac{1}{a}\frac{1}{b}}} = \sqrt{ab} = Y.$$

11 The answer is $\boxed{48}$ Let $[a, b]$ denote the time interval from a to b minutes. Someone had to observe the worm during $[0, 1]$; the worm thus traveled 12 inches during this time. Likewise, someone observed the worm during $[2, 3]$, and the worm again traveled 12 inches during this final minute. There is a gap of one minute (the interval $[1, 2]$) during which the worm traveled at least 12 inches. Could it have traveled more? Yes: consider the two overlapping observation intervals $I = [1 - \varepsilon, 2 - \varepsilon]$ and $J = [1 + \varepsilon, 2 + \varepsilon]$, where ε is a small (positive) number. If the worm had already traveled its first 12 inches by time $1 - \varepsilon$, then it will travel another 12 inches during the time interval I. Moreover, it is possible that the worm traveled so fast that it covered 12 inches from time $1-\varepsilon$ to time $1+\varepsilon$. Then it will travel another 12 inches during the time interval J. If the worm sped up during J so that it completed its 12 inches by time 2 minutes, then the worm will have traveled 24 inches in total during the interval $[1, 2]$. And notice that this is the maximum amount possible, for we can always isolate two overlapping intervals of length 1 minute during which the worm travels exactly 12 inches, hence the total is at most 24 inches.

Here is a distance-time graph, which should help make the above argument clearer.



This was a very tricky problem, which few people answered correctly. Here is a challenge: what would the answer be if we replaced "3 minutes" in the statement of the problem to "3 minutes, 10 seconds"?

12 The answer is $\boxed{29}$ Transform the equation into something that factors, by multiplying both sides by $x + y$ and adding 99^2. This yields $xy - 99x - 99y + 99^2 = 99^2$, or

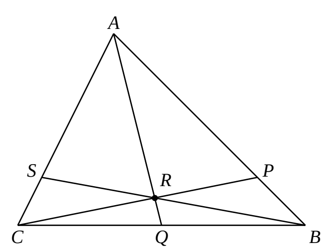
$$(x - 99)(y - 99) = 99^2.$$

For any divisor d of 99^2, we can write $x - 99 = d, y - 99 = 99^2/d$, which will yield an ordered pair (x, y) of integers that satisfy the above equation, and moreover, each solution (x, y) yields a different divisor d. Since $99^2 = 3^4 \cdot 11^2$, there are $(4 + 1)(2 + 1) = 15$ positive divisors of 99^2 and hence 30 integral divisors. Unfortunately, one of these divisors is $d = -99$, which corresponds to the values $x = 0, y = 0$. These are a solution to $(x - 99)(y - 99) = 99^2$, but not to the original equation, since you cannot divide by zero. That's why the answer is 29, instead of 30.

13 The answer is $\boxed{6}$ We shall use lLtriangles share the same vertex and their bases lie on the same line, then the ratio of the areas of the triangles equals the ratio of the base lengths. (This follows because the two triangles have the same altitude.) Draw a line from B through R, meeting side AC at point S. Let $x = AR/RQ$, and without loss of generality, let $[BPR] = 1$. By the principle mentioned above, since $AP/PB = 3$, we have $[ARP] = 3$. Likewise, $[BRQ] = [ARB]/x = 4/x$. Since $BQ/QC = 1$, we have $[CQR] = [BRQ]$, and $[ACQ] = [ABQ]$, which implies that $[ASR] = 3, [RSC] = 1$, and $[CQR] = 4/x$. Employing the principle for the final time, we have $[SBC] = [ABS]/3$, which leads to the equation

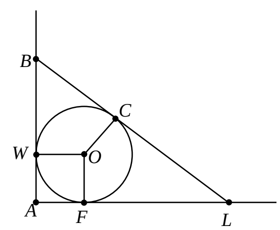
$$1 + \frac{4}{x} + \frac{4}{x} = \frac{1+3+3}{3},$$

and $x = 6$.



The second solution uses the almost-magical folklore method called **weights**. We will present the solution, but you will need to figure out why it works: Place a weight of 1 at A. Since $AP/PB = 3$, we balance it with a weight of 3 at B. This is equivalent to a weight of $1 + 3 = 4$ at P. Likewise, we balance the weight of 3 at B with a weight of 3 at C and place $3 + 3 = 6$ at Q. In order for the weights at A and Q to balance, we must have $AR/RQ = 6$.

14 The answer is $\boxed{3}$ The problem is equivalent to finding the length of side AB below, where the inscribed circle of triangle ABL had a radius of 1. The quickest way to solve this problem is by experience: many of you should already know the nifty fact (try to prove it on your own) that the radius of in the inscribed circle (also known as the **inradius**) of the a 3-4-5 right triangle is 1. But without this hindsight, it is still pretty easy.



Let $x = AB$. Then $BW = BC = x - 1$, and since $AF = 1$, we have $CL = FL = 4 - 1 = 3$. Then the pythagorean theorem leads to

$$x^2 + 4^2 = (x - 1 + 3)^2.$$

15 The answer is $\boxed{231}$ Each term will be of the form $Kx^a y^b z^c$, where a, b, c are non-negative integers whose sum is 20. The number of different terms is thus the number of different ordered triples of non-negative integers (a, b, c) such that $a + b + c = 20$. There are several ways of enumerating this. One is to use the well-known "balls in boxes" or "stars and bars" formula (see any book on combinatorics or discrete mathematics or *The Art and Craft of Problem Solving*) and this yields $\binom{20+3-1}{20} = \binom{22}{2}$. Another method is to fix $a \geq 0$, then notice that there are $20 - a + 1$ ordered pairs (b, c) of nonnegative integers such that $b + c = 20 - a$. Hence the number of triples is the sum

$$(20 - 0 + 1) + (20 - 1 + 1) + \cdots (20 - 20 + 1).$$

16 The answer is $\boxed{8}$ Observe that the quantity to be minimized is just the square of the distance between the two points $(u, \sqrt{2 - u^2})$ and $(v, 9/v)$. This is the distance between an arbitrary point on a the circle $x^2 + y^2 = 2$ and a point on the upper half of the hyperbola $y = 9/x$. Thus we need to find the minimum distance between these two curves. It is easy to see (draw a picture) that this attained by picking the point (1, 1) on the circle and the point (3, 3) on the hyperbola.

17 The answer is $\boxed{-112}$ The easiest way to do this is if you already know (and you should!) the factorization

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz).$$

(See p. 78 of *The Art and Craft of Problem Solving* for an explanation of how this factorization can be "discovered.") At this point, the only quantity that we need to compute is $xy + yz + xz$, but that is easy to extract from the identity (that you *really* should know!)

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + xz),$$

concluding that $xy + yz + xz = 0$.

Alternatively, if you did not know the exotic factorization above, you could proceed as follows: First compute $xy + yz + xz = 0$ as above, and then write $x + y = 7 - z$, and then cube both sides (now we are cubing binomials, which is easier), using the (essential!) identity

$$(a + b)^3 = a^3 + b^3 + 3ab(a + b).$$

We have

$$x^3 + y^3 + 3xy(x + y) = 7^3 - z^3 - 21z(7 - z).$$

Now, substitute $7 - z$ for $x + y$ in the left side and $x + y$ for $7 - z$ in the right, and we get

$$x^3 + y^3 + 21xy - 3xyz = 7^3 - z^3 - 21zx - 21zy.$$

Now we can solve for xyz, since we know that $x^3 + y^3 + z^3 = 7$ and $xy + yz + xz = 0$.

18 The answer is $\boxed{x = 2^{\sqrt[3]{4}}}$ Let $w = \log_x y$. Then $x^{1/w} = 2$, so $x = 2^w$ and $1 = w \log_x 2$. Also, $y^w = 16$. Taking logs base x, we have $w \log_x y = \log_x 16$, or $w^2 = \log_x 16 = 4 \log_x 2$. Since $1 = w \log_x 2$, we have $w^3 = 4$, hence $w = \sqrt[3]{4}$ and $x = 2^w = 2^{\sqrt[3]{4}}$.

19 The answer is $\boxed{2 \csc 9^\circ}$ Let the center of the circle be O. And let $2\alpha = \angle A_1 O A_2 = 18^\circ$. Then (draw a picture!) $A_1 A_2 = 2 \sin \alpha$. Likewise, the next diagonal which is parallel to $A_1 A_2$ is $A_{20} A_3$ and the length of this diagonal is $2 \sin 3\alpha$. The sum in question is

$$S = 2(\sin \alpha + \sin 3\alpha + \sin 5\alpha + \cdots + \sin 19\alpha).$$

There are many ways to evaluate this sum. One is by memorizing the formulas (consult any standard reference, such as the *CRC Standard Math Tables* handbook), not a bad idea, but not much fun. Another method uses complex numbers and is well worth trying. (See Chapter 1 of *Visual Complex Analysis* by Tristan Needham or Chapter 4 of *The Art and Craft of Problem Solving* for ideas.) We will illustrate perhaps the easiest method, which uses telescoping sums. Multiply S by $\sin \alpha$, getting

$$S \sin \alpha = 2(\sin \alpha \sin \alpha + \sin 3\alpha \sin \alpha + \sin 5\alpha \sin \alpha + \cdots + \sin 19\alpha \sin \alpha).$$

Now apply the identity

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$

to this sum, and we get

$$\begin{aligned} S \sin\alpha &= (\cos 0 - \cos 2\alpha) + (\cos 2\alpha - \cos 4\alpha) + \cdot + (\cos 18\alpha - \cos 20\alpha) \\ &= \cos 0 - \cos 20\alpha \\ &= 1 - (-1) \\ &= 2. \end{aligned}$$

20 The answer is $\boxed{7/128}$ This was an incredibly difficult question which probably no one got (three people chose the correct answer but I suspect this was just lucky guessing). Here is a sketch of the solution, which asserts that for n points, the probability is $(n^2 - n + 2)/2^n$. To really understand the problem, you will need to fill in many details.

- Fix n points $p_1, p_2, \ldots, p_n$ placed randomly on the sphere. Each point has an **antipode** which is directly opposite it (i.e., if one point is at the north pole, its antipode is at the south pole). Let us denote the antipode of the point p by $-p$. The n points naturally give rise to a "sorority" of 2^n different sets of n points, namely the sets $\{a_1 p_1, a_2 p_2, \ldots, a_n p_n\}$, where each a_i is either $+1$ or -1. (Assume that none of the points are antipodes of one another, so these 2^n sets are all different.)
- Let us look at these 2^n sets and try to count how many of them lie on one hemisphere. We will use the very useful observation that a pair of antipodes p and $-p$ determine a great circle, namely the "equator" which is perpendicular to the line joining p and $-p$ (which goes through the center of the sphere). So now, for the moment, forget about the n points, and replace them with the corresponding n great circles.
- These n great circles divide the sphere into $n^2 - n + 2$ different regions (to see why, look at problem #20 of the 1997 Test of Ingenuity, which essentially asked this question).
- Each of these $n^2 - n + 2$ regions correspond to different choices of points/antipodes. For example, suppose that we distinguish a direction that we call "North" (N) and its antipode is of course, "South" (S). Then given any great circle, one of its defining points is on the northern side and one is on the southern side. So each of these $n^2 - n + 2$ regions can be coded with a string of n letters such as NSSS...N. Although there are 2^n possible codes, there are not 2^n different regions—for $n \geq 3$, there are fewer regions than codes. Each region corresponds to a different code.
- Each region certainly lies in a single hemisphere, and conversely if a set of n points (chosen from the 2^n possible sets) lies in a single hemisphere, these n points would lie in one of the regions.
- Hence there is a one-to-one correspondence between regions and sets of points lying in one hemisphere, so there will be $n^2 - n + 2$ sets lying in one hemisphere.

- All of the infinitely many choice of n random points on a sphere (except for a few special cases which have zero probability) can be partitioned into sororities of 2^n sets, and within each sorority, the probability that a set lies in one hemisphere is $(n^2 - n + 2)/2^n$. Therefore the desired probability will be $(n^2 - n + 2)/2^n$.

To test your understanding, see how this problem can be used to solve problem A6 on the 1992 Putnam exam.